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ASYMPTOTIC BEHAVIOR OF THE LOGARITHMS OF ENTIRE FUNCTIONS OF IMPROVED REGULAR GROWTH IN THE METRIC OF $L^q[0, 2\pi]$

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We describe the asymptotic behavior of the logarithms of entire functions of improved regular growth with zeros on a finite system of rays in the metric of $L^q[0, 2\pi]$.

1. Introduction and Formulation of the Main Results

One of the main problems in the theory of entire functions is the investigation of the relationship between the regularity of growth of a function and the distribution of its zeros. At the end of the 1930s, the investigations of this problem carried out by Levin and Pfluger [1] (see also [2, 3]) led to the formation of the theory of entire functions of completely regular growth. The entire functions of completely regular growth are characterized by the regular behavior not only of their modulus but also of their argument. Note that it is important to have different criteria of belonging of the entire functions to the class of completely regular growth. There are numerous necessary and sufficient conditions of completely regular growth for the entire functions of positive order (see [2, 3]). Thus, in particular, Azarin [5] obtained a criterion of regularity of this kind in terms of the Fourier coefficients. At the same time, in [6, p. 78], Kondratyuk established the corresponding criterion in terms of the q-norm of the logarithm of modulus of an entire function in the space $L^q[0, 2\pi]$.

The Fourier method plays an important role in the development of the theory of entire functions of completely regular growth. Its systematic application was originated in the works by Rubel and Taylor (see [4]). In particular, by using this method, Kondratyuk [6, p. 78; 7] and Vasyl'kiv [8, 9] described the property of completely regular growth of the logarithm of modulus and argument of entire and meromorphic functions of positive order in the metric of $L^q[0, 2\pi]$. Similar results for entire functions of order zero were obtained in [10, 11].

The notion of entire function of improved regular growth was introduced in [12, 13]. Moreover, in the same works, the authors also obtained some criteria of this regularity in terms of the distribution of zeros located on a finite system of rays. In [14], this notion was generalized to the case of subharmonic functions. In [15], a criterion of improved growth of entire functions of positive order with zeros on a finite system of rays was established in terms of their Fourier coefficients. In [16], we described the improved regularity of growth of the logarithm of modulus for the entire functions of positive order with zeros on a finite system of rays in the metric of $L^q[0, 2\pi]$. In the general case (for an arbitrary distribution of zeros), the asymptotic behavior of entire functions of improved regular growth was investigated in [14, 17–19]. However, the behavior of the arguments of entire functions of improved regular growth was not investigated.

An entire function f is called a function of improved regular growth [12, 13] if, for some $\rho \in (0, +\infty)$ and $\rho_1 \in (0, \rho)$ and a 2π-periodic ρ-trigonometrically convex function $h \not\equiv -\infty$, there exists a set $U \subset \mathbb{C}$ contained in a union of disks with finite sum of radii such that

$$\log|f(z)| = |z|^\rho h(\arg z) + o(|z|^{\rho_1}), \quad U \not\ni z = re^{i\varphi} \to \infty.$$

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If an entire function f is a function of improved regular growth, then it has the order ρ and the indicator $h(\varphi)$ [12]. The function $h(\varphi)$ has the right derivative, which is continuous everywhere, except an at most countable set [6, pp. 93, 94, 110; 9, p. 138] (see also [1, pp. 76–78, 199]).

Let f be an entire function, $f(0) = 1$ and let (λ_n) be a sequence of its zeros. Assume that the function

$$\log f(z) = \int_0^z \frac{f'(\zeta)}{f(\zeta)}\, d\zeta$$

is defined in the complex plane with radial cuts made from zeros of the entire function f to infinity. The function $\log f(z)$ is a single-valued branch of the multivalued function

$$\operatorname{Log} f(z) = \log|f(z)| + i \operatorname{Arg} f(z)$$

such that $\log f(0) = 0$. Let

$$c_k(r, \log|f|) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\varphi}\log|f(re^{i\varphi})|\, d\varphi, \qquad k \in \mathbb{Z}, \quad r > 0,$$

be the Fourier coefficients of the function $\log|f(re^{i\varphi})|$ and let

$$n(t,\psi; f) = \sum_{|\lambda_n|\le t,\, \arg\lambda_n = \psi} 1.$$

It is known that, by the Hadamard–Borel theorem [1, p. 38; 4, 6, 12, 13], an entire function f, $f(0) = 1$, of order $\rho \in (0, +\infty)$ has the form

$$f(z) = e^{Q(z)}\prod_{n=1}^{\infty} E\left(\frac{z}{\lambda_n}, p\right),$$

where $\lambda_n \neq 0$ are zeros of the function $f(z)$,

$$Q(z) = \sum_{k=1}^{\nu} Q_k z^k$$

is a polynomial of degree $\nu \le \rho$, $p \le \rho$ is the least integer for which

$$\sum_{n\in\mathbb{N}} |\lambda_n|^{-p-1} < +\infty,$$

and

$$E(w,p) = (1-w)\exp(w + w^2/2 + \ldots + w^p/p)$$

is a primary Weierstrass factor of the pth kind.

The following theorem gives necessary and sufficient conditions for the improved regular growth of entire functions of positive order with zeros on a finite system of rays:

Theorem A. *Suppose that f is an entire function of order $\rho \in (0, +\infty)$ with zeros on a finite system of rays $\{z\colon \arg z = \psi_j\}$, $j \in \{1, \dots, m\}$, $0 \le \psi_1 < \psi_2 < \dots < \psi_m < 2\pi$. Then the following assertions are equivalent:*

(i) for some $\rho_2 \in (0, \rho)$ and each $j \in \{1, \dots, m\}$, the following relation is true:

$$n(t, \psi_j; f) = \Delta_j t^\rho + o(t^{\rho_2}), \qquad t \to +\infty, \quad \Delta_j \in [0, +\infty), \tag{1}$$

and, in addition, for integer ρ and some $\rho_3 \in (0, \rho)$ and $\delta_f \in \mathbb{C}$,

$$\sum_{0<|\lambda_n|\le r} \lambda_n^{-\rho} = \delta_f + o(r^{\rho_3 - \rho}), \quad r \to +\infty; \tag{2}$$

(ii) f is a function of improved regular growth with indicator $h(\varphi)$; moreover, if ρ is a noninteger number, then

$$h(\varphi) = \sum_{j=1}^{m} h_j(\varphi), \tag{3}$$

where $h_j(\varphi)$ is a 2π-periodic function given by the equality

$$h_j(\varphi) = \frac{\pi \Delta_j}{\sin \pi\rho} \cos \rho(\varphi - \psi_j - \pi)$$

in the interval $[\psi_j, \psi_j + 2\pi)$; for $\rho \in \mathbb{N}$, the following relation is true:

$$h(\varphi) = \begin{cases} \tau_f \cos(\rho\varphi + \theta_f) + \sum_{j=1}^{m} h_j(\varphi), & \rho = p, \\ Q_\rho \cos \rho\varphi, & \rho = p + 1, \end{cases} \tag{4}$$

where $\tau_f = |\delta_f/\rho + Q_\rho|$, $\theta_f = \arg(\delta_f/\rho + Q_\rho)$, and $h_j(\varphi)$ is a 2π-periodic function given by the equality

$$h_j(\varphi) = \Delta_j(\pi - \varphi + \psi_j) \sin \rho(\varphi - \psi_j) - \frac{\Delta_j}{\rho} \cos \rho(\varphi - \psi_j)$$

in the interval $[\psi_j, \psi_j + 2\pi)$;

(iii) for some $\rho_4 \in (0, \rho)$ and $k_0 \in \mathbb{Z}$ and each $k \in \{k_0, k_0 + 1, \dots, k_0 + m - 1\}$,

$$c_k(r, \log|f|) = c_k r^\rho + o(r^{\rho_4}) \quad \text{as} \quad r \to +\infty, \qquad c_k := \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\varphi} h(\varphi)\, d\varphi;$$

(iv) for some $\rho_5 \in (0, \rho)$ and every $q \in [1, +\infty)$, the following equality is true:

$$\left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\log|f(re^{i\varphi})|}{r^\rho} - h(\varphi) \right|^q d\varphi \right\}^{1/q} = o(r^{\rho_5 - \rho}), \quad r \to +\infty.$$

The equivalence of the first and second assertions was proved in [12, 13]. The equivalence of the second and third assertions was established in [15]. The equivalence of the second and fourth assertions was demonstrated in [16].

The problem of finding new criteria of improved regular growth for entire functions of positive order seems to be quite urgent. The aim of the present paper is to analyze the asymptotic behavior of logarithms of the entire functions of improved regular growth with zeros on a finite system of rays in the metrics of the spaces $L^q[0, 2\pi]$, $q \in [1, +\infty)$.

Theorem 1. *Suppose that f is an entire function of order $\rho \in (0, +\infty)$ with zeros on a finite system of rays $\{z\colon \arg z = \psi_j\}$, $j \in \{1, \ldots, m\}$, $0 \le \psi_1 < \psi_2 < \ldots < \psi_m < 2\pi$. For the function f to be a function of improved regular growth, it is necessary and sufficient that the following relation be true for some $\rho_6 \in (0, \rho)$ and every $q \in [1, +\infty)$:*

$$\left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\arg f(re^{i\varphi})}{r^\rho} + \frac{h'(\varphi)}{\rho} \right|^q d\varphi \right\}^{1/q} = o(r^{\rho_6 - \rho}), \quad r \to +\infty, \tag{5}$$

where $h(\varphi)$ is the indicator of f.

Corollary 1. *Let f be an entire function of order $\rho \in (0, +\infty)$ with zeros on a finite system of rays $\{z\colon \arg z = \psi_j\}$, $j \in \{1, \ldots, m\}$, $0 \le \psi_1 < \psi_2 < \ldots < \psi_m < 2\pi$. For the function f to be a function of improved regular growth, it is necessary and sufficient that the following relation be true for some $\rho_7 \in (0, \rho)$ and any $q \in [1, +\infty)$:*

$$\left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\log f(re^{i\varphi})}{r^\rho} - \widetilde{h}(\varphi) \right|^q d\varphi \right\}^{1/q} = o(r^{\rho_7 - \rho}), \quad r \to +\infty,$$

where

$$\widetilde{h}(\varphi) = h(\varphi) - i\frac{h'(\varphi)}{\rho}.$$

2. Auxiliary Statements

Let f be an entire function, $f(0) = 1$, and let (λ_n) be the sequence of its zeros. For $k \in \mathbb{Z}$ and $r > 0$, we denote

$$N(r, \psi; f) = \int_0^r \frac{n(t, \psi; f)}{t}\, dt, \qquad N^*(r, \psi; f) = \int_0^r \frac{N(t, \psi; f)}{t}\, dt,$$

$$c_k(r, \arg f) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\varphi} \arg f(re^{i\varphi})\, d\varphi,$$

$$n_k(r, f) = \sum_{|\lambda_n| \le r} e^{-ik \arg \lambda_n}, \qquad N_k(r, f) = \int_0^r \frac{n_k(t, f)}{t}\, dt, \qquad N_k^*(r, f) = \int_0^r \frac{N_k(t, f)}{t}\, dt.$$

Lemma 1. *If an entire function f of order $\rho\in(0,+\infty)$ with zeros on a finite system of rays $\{z:\arg z=\psi_j\}$, $j\in\{1,\dots,m\}$, $0\le\psi_1<\psi_2<\dots<\psi_m<2\pi$, is a function of improved regular growth, then, for some $\rho_8\in(0,\rho)$,*

$$c_k(r,\arg f)=-ikc_k\frac{r^\rho}{\rho}+\frac{k}{k^2+1}o(r^{\rho_8}),\quad r\to+\infty, \tag{6}$$

regularly in $k\in\mathbb{Z}$, where

$$c_k=\frac{1}{2\pi}\int\limits_0^{2\pi}e^{-ik\varphi}h(\varphi)\,d\varphi=\frac{\rho}{\rho^2-k^2}\sum_{j=1}^m\Delta_j e^{-ik\psi_j},\quad \Delta_j\in[0,+\infty), \tag{7}$$

if ρ is a noninteger number and, for $\rho\in\mathbb{N}$,

$$c_k=\begin{cases}\dfrac{\rho}{\rho^2-k^2}\sum_{j=1}^m\Delta_j e^{-ik\psi_j}, & |k|\ne\rho=p,\\[2mm] \dfrac{\tau_f e^{i\theta_f}}{2}-\dfrac{1}{4\rho}\sum_{j=1}^m\Delta_j e^{-i\rho\psi_j}, & k=\rho=p,\\[2mm] 0, & |k|\ne\rho=p+1,\\[2mm] \dfrac{Q_\rho}{2}, & k=\rho=p+1.\end{cases} \tag{8}$$

Proof. If the conditions of Lemma 1 are satisfied, then [20, p. 10] (Lemma 1) (see also [16]), for some $\rho_8\in(0,\rho)$, the relation

$$c_k(r,\log|f|)=c_kr^\rho+\frac{o(r^{\rho_8})}{k^2+1},\quad r\to+\infty,$$

where c_k is given by relations (7) and (8), holds uniformly in $k\in\mathbb{Z}$. Since [8, p. 43]

$$c_k(r,\arg f)=-ik\int\limits_0^r\frac{c_k(t,\log|f|)}{t}\,dt,\quad k\in\mathbb{Z},$$

we obtain (6) from the last relation.

Lemma 1 is proved.

Lemma 2. *Suppose that f is an entire function of order $\rho\in(0,+\infty)$ with zeros on a finite system of rays $\{z:\ \arg z=\psi_j\}$, $j\in\{1,\dots,m\}$, $0\le\psi_1<\psi_2<\dots<\psi_m<2\pi$, and the asymptotic equality (5) with function $h(\varphi)$ given by relations (3) and (4) is true for some $\rho_6\in(0,\rho)$ and any $q\in[1,+\infty)$. Then, for all $k\in\mathbb{Z}\setminus\{0\}$,*

$$c_k(r,\arg f)=-ikc_k\frac{r^\rho}{\rho}+o(r^{\rho_6}),\quad r\to+\infty, \tag{9}$$

$$N_k^*(r,f)=c_k\left(1-\frac{k^2}{\rho^2}\right)\frac{r^\rho}{\rho}+o(r^{\rho_6}),\quad r\to+\infty, \tag{10}$$

where c_k are given by relations (7) and (8).

Proof. Under the conditions of Lemma 2, for all $k \in \mathbb{Z} \setminus \{0\}$, we find

$$\left| \frac{c_k(r, \arg f)}{r^\rho} + \frac{ik}{\rho} c_k \right| \le \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\arg f(re^{i\varphi})}{r^\rho} + \frac{h'(\varphi)}{\rho} \right| d\varphi$$

$$\le \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\arg f(re^{i\varphi})}{r^\rho} + \frac{h'(\varphi)}{\rho} \right|^q d\varphi \right\}^{1/q} = o(r^{\rho_6 - \rho}), \quad r \to +\infty.$$

Thus, for some $\rho_6 \in (0, \rho)$ and all $k \in \mathbb{Z} \setminus \{0\}$, relation (9) is true with c_k given by relations (7) and (8). Since [8, p. 43]

$$N_k^*(r, f) = \frac{i}{k} c_k(r, \arg f) - ik \int_0^r \frac{dt}{t} \int_0^t \frac{c_k(u, \arg f)}{u} du, \qquad k \in \mathbb{Z} \setminus \{0\}, \quad r > 0,$$

by virtue of (9), for all $k \in \mathbb{Z} \setminus \{0\}$, we obtain

$$N_k^*(r, f) = c_k \frac{r^\rho}{\rho} + o(r^{\rho_6}) - ik \int_0^r \frac{dt}{t} \int_0^t \left(-\frac{ik}{\rho} c_k u^{\rho - 1} + o(u^{\rho_6 - 1}) \right) du$$

$$= c_k \left(1 - \frac{k^2}{\rho^2} \right) \frac{r^\rho}{\rho} + o(r^{\rho_6}), \quad r \to +\infty.$$

Therefore, equality (10) is also true.

Lemma 2 is proved.

Lemma 3. *Suppose that f is an entire function of order $\rho \in (0, +\infty)$ with zeros on a finite system of rays $\{z\colon \arg z = \psi_j\}$, $j \in \{1, \ldots, m\}$, $0 \le \psi_1 < \psi_2 < \ldots < \psi_m < 2\pi$. In order that the equality*

$$N^*(r, \psi_j; f) = \frac{\Delta_j}{\rho^2} r^\rho + o(r^{\rho_6}), \quad r \to +\infty, \quad \Delta_j \in [0, +\infty), \tag{11}$$

be true for some $\rho_6 \in (0, \rho)$ and each $j \in \{1, \ldots, m\}$, it is necessary and sufficient that relation (10) hold with c_k given by relations (7) and (8) for some $\rho_6 \in (0, \rho)$ and $k_0 \in \mathbb{Z} \setminus \{0\}$ and each $k \in \{k_0, k_0 + 1, \ldots, k_0 + m - 1\}$ and, moreover,

$$\sum_{j=1}^m \Delta_j e^{-i\rho\psi_j} = 0$$

for $\rho \in \mathbb{N}$.

Lemma 3 is proved by using the equalities

$$N_k^*(r, f) = \sum_{j=1}^m e^{-ik\psi_j} N^*(r, \psi_j; f), \qquad k \in \mathbb{Z},$$

in exactly the same way as Lemma 5 in [15, p. 1720].

Lemma 4. *Suppose that* $\rho\in(0,+\infty)$*. In order that equality (1) be true for some* $\rho_2\in(0,\rho)$ *and each* $j\in\{1,\dots,m\}$*, it is necessary and sufficient that inequality (11) hold for some* $\rho_6\in(0,\rho)$ *and each* $j\in\{1,\dots,m\}$*.*

The proof of Lemma 4 is contained in the proof of Lemma 3 in [22, p. 143] (see also [12, 13, 15]).

Lemma 5. *If an entire function* f *of order* $\rho\in\mathbb{N}$ *satisfies the conditions of Lemma 2, then condition (2) is satisfied with* $\delta_f=\rho\big(\tau_f e^{i\theta_f}-Q_\rho\big)$ *for some* $\rho_3\in(0,\rho)$*.*

Proof. By Lemma 2, for some $\rho_6\in(0,\rho)$ and each $k\in\mathbb{Z}\setminus\{0\}$, relations (9) and (10) are true with c_k given by relations (7) and (8). Moreover, by Lemmas 3 and 4, equality (1) and the relation

$$\sum_{j=1}^{m}\Delta_j e^{-i\rho\psi_j}=0$$

are true for some $\rho_2\in(0,\rho)$ and each $j\in\{1,\dots,m\}$. Since [8, p. 43]

$$ic_k(r,\arg f)=c_k(r,\log|f|)+\frac{1}{k}\sum_{0<|\lambda_n|\le r}\left(\frac{\overline{\lambda}_n}{r}\right)^k-\frac{n_k(r,f)}{k},\quad k\in\mathbb{N},$$

and [13, p. 21; 21]

$$c_\rho(r,\log|f|)=\frac{1}{2}Q_\rho r^\rho+\frac{1}{2\rho}\sum_{0<|\lambda_n|\le r}\left(\left(\frac{r}{\lambda_n}\right)^\rho-\left(\frac{\overline{\lambda}_n}{r}\right)^\rho\right),\quad k=\rho=p,$$

we get

$$ic_\rho(r,\arg f)=\frac{1}{2}Q_\rho r^\rho+\frac{1}{2\rho}\sum_{0<|\lambda_n|\le r}\left(\frac{r}{\lambda_n}\right)^\rho+\frac{1}{2\rho}\sum_{0<|\lambda_n|\le r}\left(\frac{\overline{\lambda}_n}{r}\right)^\rho-\frac{n_\rho(r,f)}{\rho},$$

where

$$n_\rho(r,f)=\sum_{j=1}^{m}e^{-i\rho\psi_j}n(r,\psi_j;f).$$

Thus, by using relations (1) and (9), we conclude that there exists $\rho_3\in(0,\rho)$ such that,

$$\begin{aligned}\sum_{0<|\lambda_n|\le r}\lambda_n^{-\rho}&=2i\rho r^{-\rho}c_\rho(r,\arg f)-\rho Q_\rho-r^{-\rho}\sum_{0<|\lambda_n|\le r}\left(\frac{\overline{\lambda}_n}{r}\right)^\rho+2r^{-\rho}n_\rho(r,f)\\
&=2i\rho r^{-\rho}c_\rho(r,\arg f)-\rho Q_\rho\\
&\quad-r^{-2\rho}\sum_{j=1}^{m}e^{-i\rho\psi_j}\int_0^r t^\rho\,dn(t,\psi_j;f)+2r^{-\rho}\sum_{j=1}^{m}e^{-i\rho\psi_j}n(r,\psi_j;f)\\
&=2i\rho r^{-\rho}c_\rho(r,\arg f)-\rho Q_\rho\\
&\quad+r^{-\rho}\sum_{j=1}^{m}e^{-i\rho\psi_j}n(r,\psi_j;f)+\rho r^{-2\rho}\sum_{j=1}^{m}e^{-i\rho\psi_j}\int_0^r t^{\rho-1}n(t,\psi_j;f)\,dt\end{aligned}$$

$$= \rho(\tau_f e^{i\theta_f} - Q_\rho) + \sum_{j=1}^{m} \Delta_j e^{-i\rho\psi_j} + o(r^{\rho_3-\rho}) = \delta_f + o(r^{\rho_3-\rho}) \quad \text{as} \quad r \to +\infty.$$

It remains to note that, for $\rho = p+1$, condition (2) follows from condition (1) [13, p. 23].

Lemma 5 is proved.

3. Proofs of Theorem 1 and Corollary 1

We first prove Theorem 1. *Necessity.* Let f be an entire function of regular growth of order $\rho \in (0, +\infty)$ with zeros on the finite system of rays $\{z\colon \arg z = \psi_j\}$, $j \in \{1, \ldots, m\}$, $0 \le \psi_1 < \psi_2 < \ldots < \psi_m < 2\pi$, and let $h(\varphi)$ be its indicator given by relations (3) and (4). By Lemma 1, for any constant $C > 0$ and any $r > r_0$, we obtain

$$\left| \frac{c_k(r, \arg f)}{r^\rho} + \frac{ik}{\rho} c_k \right| \le C \frac{k}{k^2+1}, \quad k \in \mathbb{Z}. \tag{12}$$

Hence, the sequence $\left(r^{-\rho} c_k(r, \arg f) + \frac{ik}{\rho} c_k \right)_{k \in \mathbb{Z}}$ belongs to the space $l_{\widetilde{q}}$ for all $\widetilde{q} > 1$ and $r > r_0$. By using the Hausdorff–Young theorem [6, pp. 5, 6], for $q \ge 2$, $q^{-1} + \widetilde{q}^{-1} = 1$, we obtain

$$\left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\arg f(re^{i\varphi})}{r^\rho} + \frac{h'(\varphi)}{\rho} \right|^q d\varphi \right\}^{1/q} \le \left\{ \sum_{k \in \mathbb{Z}} \left| \frac{c_k(r, \arg f)}{r^\rho} + \frac{ik}{\rho} c_k \right|^{\widetilde{q}} \right\}^{1/\widetilde{q}}.$$

In view of (12), the obtained series is uniformly convergent for all $r > r_0$. Passing to the limit in this expression as $r \to +\infty$ and using Lemma 1, we get (5) for $q \ge 2$. In view of the Hölder inequality, this yields relation (5) also for $1 \le q < 2$.

Sufficiency. The proof of sufficiency follows from Lemmas 2–5 and the equivalence of the first and second assertions in Theorem A.

Theorem 1 is proved.

Corollary 1 directly follows from Theorem 1 and the equivalence of the second and fourth assertions of Theorem A.

REFERENCES

1. B. Ya. Levin, *Distribution of Roots of Entire Functions* [in Russian], Gostekhizdat, Moscow (1956).
2. A. A. Gol'dberg, "B. Ya. Levin is a founder of the theory of functions of completely regular growth," *Mat. Fiz., Anal., Geometr.*, **1**, No. 2, 186–192 (1994).
3. A. A. Gol'dberg, B. Ya. Levin, and I. V. Ostrovskii, "Entire and meromorphic functions," in: *Itogi VINITI, Ser. Contemporary Problems of Mathematics. Fundamental Trends* [in Russian], Vol. 85, VINITI, Moscow (1991), pp. 5–186.
4. L. A. Rubel, *Entire and Meromorphic Functions*, Springer, New York (1996).
5. V. S. Azarin, "On the regularity of growth of the Fourier coefficients of the logarithm of modulus of an entire function, " *Teor. Funkts. Functs. Anal. Prilozh.*, Issue 27, 9–21 (1977).
6. A. A. Kondratyuk, *Fourier Series and Meromorphic Functions* [in Russian], Vyshcha Shkola, Lviv (1988).
7. R. Z. Kalynets' and A. A. Kondratyuk, "On the regularity of the growth of the modulus and argument of an entire function in the metric of $L^p[0, 2\pi]$," *Ukr. Mat. Zh.*, **50**, No. 7, 889–896 (1998); ***English translation:*** *Ukr. Math. J.*, **50**, No. 7, 1009–1018 (1998).
8. Ya. V. Vasyl'kiv, "Asymptotic behavior of the logarithmic derivatives and logarithms of meromorphic functions of completely regular growth in the metric of $L^p[0, 2\pi]$. Part 1," *Mat. Stud.*, **12**, No. 1, 37–58 (1999).

9. Ya. V. Vasyl'kiv, "Asymptotic behavior of the logarithmic derivatives and logarithms of meromorphic functions of completely regular growth in the metric of $L^p[0, 2\pi]$. Part 2," *Mat. Stud.*, **12**, No. 2, 135–144 (1999).
10. O. V. Bondar and M. V. Zabolots'kyi, "Criteria for the regularity of growth of the logarithm of modulus and the argument of an entire function," *Ukr. Mat. Zh.*, **62**, No. 7, 885–893 (2010); ***English translation:*** *Ukr. Math. J.*, **62**, No. 7, 1028–1039 (2010).
11. M. V. Zabolotskii and O. V. Kostyuk, "Regular growth of various characteristics of the entire functions of order zero," *Mat. Zametki*, **100**, No. 3, 363–374 (2016).
12. B. V. Vynnyts'kyi and R. V. Khats', "On the regularity of growth of an entire function of noninteger order with zeros on a finite system of rays," *Mat. Stud.*, **24**, No. 1, 31–38 (2005).
13. R. V. Khats', "On entire functions of improved regular growth of integer order with zeros on a finite system of rays," *Mat. Stud.*, **26**, No. 1, 17–24 (2006).
14. M. O. Hirnyk, "Subharmonic functions of improved regular growth," *Dop. Nats. Akad. Nauk Ukr.*, No. 4, 13–18 (2009).
15. R. V. Khats', "Regularity of growth of Fourier coefficients of entire functions of improved regular growth," *Ukr. Mat. Zh.*, **63**, No. 12, 1717–1723 (2011); ***English translation:*** *Ukr. Math. J.*, **63**, No. 12, 1953–1960 (2012).
16. R. V. Khats', "Asymptotic behavior of entire functions of improved regular growth in the metric of $L^p[0, 2\pi]$," *Carpathian Math. Publ.*, **5**, No. 2, 341–344 (2013).
17. B. V. Vynnyts'kyi and R. V. Khats', "On asymptotic properties of entire functions similar to the entire functions of completely regular growth," *Visn. Nats. Univ. Lviv. Politekh., Fiz.-Mat. Nauky*, **718**, No. 718, 5–9 (2011).
18. R. V. Khats', "Asymptotic behavior of averaging of entire functions of improved regular growth," *Carpathian Math. Publ.*, **5**, No. 1, 129–133 (2013).
19. I. E. Chyzhykov, "Pfluger-type theorem for functions of refined regular growth," *Mat. Stud.*, **47**, No. 2, 169–178 (2017).
20. R. V. Khats', "Averaging of entire functions of improved regular growth with zeros on a finite system of rays," *Visn. Nats. Univ. Lviv. Politekh., Fiz.-Mat. Nauky*, **718**, No. 718, 10–14 (2011).
21. R. V. Khats', "On the Fourier coefficients of one class of entire functions," *Mat. Stud.*, **23**, No. 1, 99–102 (2005).
22. B. V. Vynnyts'kyi and R. V. Khats', "On the asymptotic behavior of entire functions of noninteger order," *Mat. Stud.*, **21**, No. 2, 140–150 (2004).