EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 11, Number 4 (2020), 76 – 86

UNCONDITIONAL BASES OF SYSTEMS OF BESSEL FUNCTIONS

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Communicated by E.S. Smailov

Key words: interpolation problem, complete interpolating sequence, unconditional basis, Bessel function, entire function of exponential type.

AMS Mathematics Subject Classification: Primary 41A05, 30E05, 30B60, 33C10, 34B30, 41A30, 42A65; Secondary 30D10, 30D20, 44A15, 46E30.

Abstract. We find a criterion of unconditional basicity of the system $(\sqrt{x\rho_k}J_\nu(x\rho_k) : k \in \mathbb{N})$ in the space $L^2(0;1)$ where J_ν is the Bessel function of the first kind of index $\nu \geq -1/2$ and $(\rho_k : k \in \mathbb{N})$ is a sequence of distinct nonzero complex numbers.

DOI: https://doi.org/10.32523/2077-9879-2020-11-4-76-86

1 Introduction and main results

Let $\mathrm{P} = \{\rho_k : k \in \mathbb{N}\}$ be a sequence of nonzero complex numbers. Developing the results of Pavlov [21], Nikol'skii [20] and others (see [14, 16, 24]), Minkin [19] obtained a criterion of unconditional basicity of the system of exponentials $(\exp(i\rho_k t) : k \in \mathbb{N})$ in $L^2(-\pi;\pi)$. Lyubarskii and Seip [17] found another approach to the proof of this criterion. Let

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)}, \quad z = x+iy = re^{i\theta},$$

be the Bessel function of the first kind of index $\nu \in \mathbb{R}$, where Γ is the gamma function. In this paper, we will establish a criterion of unconditional basicity of the system

$$(\sqrt{x\rho_k}J_\nu(x\rho_k) : k \in \mathbb{N}) \tag{1.1}$$

in the space $L^2(0;1)$ if $\nu \geq -1/2$.

Basis properties of the systems of Bessel functions and similar systems have been studied in a number of papers (see, for instance, [2], [12], [13] and [26]–[33]). In particular, it is well known that if $\nu > -1$ and $(\rho_k : k \in \mathbb{N})$ is a sequence of positive zeros of J_ν, then [13] (see also [26], [27], [33]) system (1.1) forms a basis in $L^2(0;1)$. Another sufficient conditions of basicity of system (1.1) with $\nu \geq -1/2$ in $L^2(0;1)$ were found in [30].

Let $\lambda_k = \rho_k$, $\lambda_{-k} := -\lambda_k$, $\Lambda = \{\lambda_k : k \in \mathbb{Z}\setminus\{0\}\}$, let $L^2(X)$ be the space of all measurable functions $f : X \to \mathbb{C}$, $X \subseteq \mathbb{R}$, satisfying

$$\|f\|^2_{L^2(X)} := \int_X |f(x)|^2\, dx < +\infty,$$

let $\nu \in (-1;+\infty)$, let $L^{2,\nu}(\mathbb{R})$ be the space of all Lebesgue measurable functions f satisfying

$$\|f\|^2_{L^{2,\nu}(\mathbb{R})} := \int_{-\infty}^{+\infty} |x|^{2\nu+1}|f(x)|^2\, dx < +\infty,$$

let $PW^{2,\nu}$ be the space of all entire functions f of exponential type $\sigma \le 1$ for which

$$\|f\|^2_{PW^{2,\nu}} := \int_{-\infty}^{+\infty} |x|^{2\nu+1}|f(x)|^2\,dx < +\infty,$$

and let $PW^{2,\nu}_+$ be the subspace of even functions $f \in PW^{2,\nu}$.

Our main result is the following statement.

Theorem 1.1. *Let $\nu \ge -1/2$. System (1.1) forms an unconditionally basis in $L^2(0;1)$ if and only if the following conditions hold:*
1) $\rho_k^2 \ne \rho_j^2$ for $k \ne j$;
2) the function

$$S(z) = \prod_{k=1}^{\infty}\left(1 - \frac{z^2}{\rho_k^2}\right)$$

is an entire function of exponential type $\sigma \le 1$ and $(z^2-\rho_1^2)^{-1}S(z) \in PW^{2,\nu}_+$;

3) $\inf\left\{\prod_{\substack{\operatorname{Im}\lambda_j>0,\\ j\ne k}}\left|\frac{\lambda_k-\lambda_j}{\lambda_k-\overline{\lambda}_j}\right| : \operatorname{Im}\lambda_k > 0\right\} > 0, \quad \inf\left\{\prod_{\substack{\operatorname{Im}\lambda_j<0,\\ j\ne k}}\left|\frac{\lambda_k-\lambda_j}{\lambda_k-\overline{\lambda}_j}\right| : \operatorname{Im}\lambda_k < 0\right\} > 0;$

4) $\inf\left\{\frac{|\lambda_k-\lambda_j|}{1+|\lambda_k-\overline{\lambda}_j|} : (k;j) \in (\mathbb{Z}\setminus\{0\})^2, k \ne j\right\} > 0;$

5) the function $u(x) = F^2(x)$, $F(x) := |x|^{\nu+1/2}\dfrac{|S(x)|}{\operatorname{dist}(x;\Lambda)}$, satisfies the (continuous) (A_2) condition

$$\sup_I \left\{\left(\frac{1}{|I|}\int_I u(x)\,dx\right)\left(\frac{1}{|I|}\int_I u^{-1}(x)\,dx\right) : I \subset \mathbb{R}\right\} < +\infty,$$

where $I \subset \mathbb{R}$ is an interval of length $|I|$, and $\operatorname{dist}(x;\Lambda)$ is the distance between the element x and the set Λ.

Remark 1. The conditions 3) – 5) may be expressed in different ways (see [17]). We will discuss this in detail below.

Let $w = (w_k)$ be a sequence of positive numbers, let $l^{2,w}$ be the space of all sequences $d = (d_k)$ for which $\|d\|^2_{2,w} := \sum_k |d_k|^2 w_k < +\infty$, and let $l^{2,w}_+$ be the subspace of $l^{2,w}$ of sequences $d = (d_k : k \in \mathbb{Z}\setminus\{0\})$ such that $d_{-k} = d_k$. Interpolation problems in the spaces $PW^{2,\nu}$ have been considered in [3]–[6], [9]–[11], [16], [18] and [22]–[24]. Following Lyubarskii and Seip [17], we say that a sequence $\Lambda = \{\lambda_k : k \in \mathbb{Z}\setminus\{0\}\}$ (sequence $\mathrm{P} = \{\rho_k : k \in \mathbb{N}\}$) is a *complete interpolating sequence* for $PW^{2,\nu}$ (for $PW^{2,\nu}_+$) if for every sequence $d \in l^{2,w}$, $w_k := |\lambda_k|^{2\nu+1}(1+|\operatorname{Im}\lambda_k|)e^{-2|\operatorname{Im}\lambda_k|}$ (sequence $d \in l^{2,w}_+$, $w_k := |\rho_k|^{2\nu+1}(1+|\operatorname{Im}\rho_k|)e^{-2|\operatorname{Im}\rho_k|}$), the interpolation problem $f(\lambda_k) = d_k$, $k \in \mathbb{Z}\setminus\{0\}$ (interpolation problem $f(\rho_k) = d_k$, $k \in \mathbb{N}$) has a unique solution $f \in PW^{2,\nu}$ ($f \in PW^{2,\nu}_+$). Following [17], we also say that a sequence $w = (w_k)$ satisfies the *discrete (A_2) condition* if

$$\sup\left\{\left(\frac{1}{n}\sum_{j=k+1}^{k+n} w_j\right)\left(\frac{1}{n}\sum_{j=k+1}^{k+n} w_j^{-1}\right) : n \in \mathbb{N}, k \in \mathbb{Z}\setminus\{0\}\right\} < +\infty.$$

Theorem 1.1 will be proved by using the results of Lyubarskii and Seip ([17, 18]), and the following statement.

Theorem 1.2. *Let $\nu \ge -1/2$ and $w_k = |\rho_k|^{2\nu+1}(1+|\operatorname{Im}\rho_k|)e^{-2|\operatorname{Im}\rho_k|}$. System (1.1) forms an unconditionally basis in $L^2(0;1)$ if and only if the sequence $\mathrm{P} = \{\rho_k : k \in \mathbb{N}\}$ is a complete interpolating sequence for $PW^{2,\nu}_+$.*

2 Proof of Theorem 1.2

To prove Theorem 1.2 we need the following auxiliary statements.

Lemma 2.1. (see [3]) *Let $\nu > -1$. Then every function $f \in L^2(0;+\infty)$ can be represented in the form*

$$f(z) = \int_0^{+\infty} \sqrt{zt} J_\nu(zt) h_f(t)\, dt$$

with some function $h_f \in L^2(0;+\infty)$. Also, we have $\|f\|_{L^2(0;+\infty)} = \|h_f\|_{L^2(0;+\infty)}$ and

$$h_f(t) = \int_0^{+\infty} \sqrt{zt} J_\nu(zt) f(z)\, dz.$$

Lemma 2.2. (see [1], [8]) *Let $\nu \geq -1/2$. A function f has the representation*

$$f(z) = \int_0^1 \sqrt{tz} J_\nu(zt) \gamma_f(t)\, dt$$

with some function $\gamma_f \in L^2(0;1)$ if and only if $f \in L^2(0;+\infty)$ and $f(z) = z^{\nu+1/2} Q_f(z)$, where Q_f is an even entire function of exponential type $\sigma \leq 1$.

By c_j we denote some positive constants.

Lemma 2.3. *Let $\nu > -1$ and $(\rho_k : k \in \mathbb{N})$ be an arbitrary sequence of nonzero complex numbers. Then there exists a positive constant c_1 such that for all $k \in \mathbb{N}$*

$$e^{2|\operatorname{Im}\rho_k|}(1 + |\operatorname{Im}\rho_k|)^{-1}/c_1 \leq \|\sqrt{t\rho_k} J_\nu(t\rho_k)\|^2_{L^2(0;1)} \leq c_1 e^{2|\operatorname{Im}\rho_k|}(1 + |\operatorname{Im}\rho_k|)^{-1}.$$

Proof. In fact, the right-hand side of this inequality follows from the following estimate (see [13], [29], [33])

$$|\sqrt{z} J_\nu(z)| \leq c_2 e^{|\operatorname{Im} z|} \left(\frac{|z|}{1+|z|}\right)^{\nu+1/2}, \quad z \in \mathbb{C}.$$

Let us prove the left-hand side of this inequality (in the case $\operatorname{Im}\rho_k = 0$ the proof is given in [26, p. 227]). Without lost of generality, we may assume that $|\rho_k| > 1$. Using relations (see [26], [27], [33])

$$J_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(\nu+1)} + O(z^{\nu+2}), \quad z \to 0,$$

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{2}\nu - \frac{\pi}{4}\right) + O\left(\frac{e^{|\operatorname{Im} z|}}{|z|^{3/2}}\right), \quad z \to \infty, \quad |\arg z| < \pi,$$

we get

$$\begin{aligned}
\|\sqrt{t\rho_k} J_\nu(t\rho_k)\|_{L^2(0;1)} &= \left(\int_0^1 |\rho_k| t |J_\nu(t|\rho_k| e^{i\theta_k})|^2\, dt\right)^{1/2} = \left(\frac{1}{|\rho_k|}\int_0^{|\rho_k|} t |J_\nu(te^{i\theta_k})|^2\, dt\right)^{1/2} \\
&= \left(\frac{1}{|\rho_k|}\int_0^1 t|J_\nu(te^{i\theta_k})|^2\, dt + \frac{1}{|\rho_k|}\int_1^{|\rho_k|} t|J_\nu(te^{i\theta_k})|^2\, dt\right)^{1/2} \\
&\geq \frac{1}{\sqrt{|\rho_k|}}\left(\int_1^{|\rho_k|} t|J_\nu(te^{i\theta_k})|^2\, dt\right)^{1/2} = \frac{1}{\sqrt{|\rho_k|}}\|\sqrt{t} J_\nu(te^{i\theta_k})\|_{L^2(1;|\rho_k|)} \\
&\geq \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{|\rho_k|}} \left\|\cos\left(te^{i\theta_k} - \frac{\pi}{2}\nu - \frac{\pi}{4}\right)\right\|_{L^2(1;|\rho_k|)} \\
&\quad - \frac{1}{\sqrt{|\rho_k|}} \left\|\frac{e^{t|\sin\theta_k|}}{t}\right\|_{L^2(1;|\rho_k|)}, \quad \rho_k := |\rho_k| e^{i\theta_k}.
\end{aligned}$$

In addition, (here and in what follows the sign $\asymp$ means that the ratio of the two sides lies between two positive constants)

$$\left\| \frac{e^{t|\sin\theta_k|}}{t} \right\|^2_{L^2(1;|\rho_k|)} \asymp \frac{e^{2|\rho_k||\sin\theta_k|}}{2|\rho_k|(1+|\rho_k||\sin\theta_k|)}, \qquad k\to\infty,$$

$$|\cos(x+iy)|^2 = \cos^2 x + \sinh^2 y,$$

$$\int_1^{|\rho_k|} \sinh^2(t\sin\theta_k)\,dt = \frac{\sinh(2|\rho_k|\sin\theta_k) - \sinh(2\sin\theta_k)}{4\sin\theta_k} - \frac{|\rho_k|-1}{2},$$

$$\int_1^{|\rho_k|} \cos^2\left(t\cos\theta_k - \frac{\pi}{2}\nu - \frac{\pi}{4}\right)dt$$
$$= \frac{|\rho_k|-1}{2} + \frac{\sin(|\rho_k|\cos\theta_k - \cos\theta_k)\sin(|\rho_k|\cos\theta_k + \cos\theta_k - \pi\nu)}{2\cos\theta_k}.$$

□

Proof of Theorem 1.2. The sequence (e_k) of nonzero elements of a separable Hilbert space H with the inner product $\langle\cdot;\cdot\rangle$ is an unconditional basis of H (see, for example, [25]) if and only if for each sequence (b_k) satisfying $\sum_k |b_k|^2\|e_k\|^{-2} < +\infty$ there is a unique element $g \in H$ such that $\langle g; e_k\rangle = b_k$ for all $k \in \mathbb{N}$. If $H = L^2(0;1)$, $e_k = \sqrt{\rho_k x}J_\nu(\rho_k x)$ and $g \in L^2(0;1)$, then by Lemmas 2.1 and 2.2, we have

$$\langle g; e_k\rangle = \int_0^1 \sqrt{\rho_k t}J_\nu(\rho_k t)\overline{g(t)}dt = \rho_k^{\nu+1/2}Q_f(\rho_k),$$

where $Q_f \in PW_+^{2,\nu}$. Therefore, using Lemma 2.3, we obtain the required statement. □

3 Proof of Theorem 1.1

Theorem 1.1 follows from Theorem 1.2 and the results of Lyubarskii and Seip ([17], [18]). So, we are going to sketch the proof of Theorem 1.1.

Let $-\infty \le a < b \le +\infty$ and $\mathbb{C}_{a,b} = \{z : a < \operatorname{Im} z < b\}$, and let $H^2(\mathbb{C}_{a,b})$ be the space of all functions f that are holomorphic in the strip $\mathbb{C}_{a,b}$ and satisfy

$$\sup\left\{\int_{-\infty}^{+\infty} |f(x+iy)|^2\,dx : a < y < b\right\} < +\infty.$$

If at least one of the numbers a or b is finite, then every function $f \in H^2(\mathbb{C}_{a,b})$ has angular limit values $f \in L^2(\partial\mathbb{C}_{a,b})$ almost everywhere on $\partial\mathbb{C}_{a,b}$, and the equality $\|f\|^2 := \int_{\partial\mathbb{C}_{a,b}} |f(z)|^2\,|dz|$ determines a norm on $H^2(\mathbb{C}_{a,b})$ (see [3], [15]). If $a \in \mathbb{R}$ and $b = +\infty$, then $H^2(\mathbb{C}_{a,b})$ is the Hardy space in the half-plane $\mathbb{C}_{a,b}$. The same can be said when $a = -\infty$ and $b \in \mathbb{R}$.

Lemma 3.1. (see [3]–[6], [22], [23]) *If $\nu \ge -1/2$ and $f \in PW^{2,\nu}$, then for any $a \in \mathbb{R}$ the function $f_+(z) = (z+ia)^{\nu+1/2}e^{iz}f(z)$ belongs to $H^2(\mathbb{C}_{-a,+\infty})$ and*

$$\int_{-\infty}^{+\infty} |x+iy+ia|^{2\nu+1}|f(x+iy+ia)|^2\,dx \le e^{2y}\int_{-\infty}^{+\infty} |x+ia|^{2\nu+1}|f(x+ia)|^2\,dx, \quad y > -a.$$

Moreover, the function $f_-(z) = (z-ia)^{\nu+1/2}e^{-iz}f(z)$ belongs to $H^2(\mathbb{C}_{-\infty,a})$ and

$$\int_{-\infty}^{+\infty} |x+iy-ia|^{2\nu+1}|f(x+iy-ia)|^2\,dx \le e^{-2y}\int_{-\infty}^{+\infty} |x-ia|^{2\nu+1}|f(x-ia)|^2\,dx, \quad y < a.$$

Lemma 3.2. (see [3]–[6], [22], [23]) *Let $\nu \geq -1/2$, $\gamma \in \mathbb{R}$ and $\beta \in \mathbb{R}$. The space (as a set) $PW_+^{2,\nu}$ coincides with the set $W^{2,\nu}[\gamma;\beta]$ of all even entire functions f of exponential type $\sigma \leq 1$ satisfying*

$$\|f\|^2_{W^{2,\nu}[\gamma;\beta]} := \int_{-\infty}^{+\infty} |x+i\gamma|^{2\nu+1}|f(x+i\beta)|^2\,dx < +\infty.$$

Moreover, the norms $\|f\|_{W^{2,\nu}[\gamma;\beta]}$ and $\|f\|_{PW_+^{2,\nu}}$ are equivalent,

$$\int_{-\infty}^{+\infty} |x|^{2\nu+1}|f(x+iy)|^2\,dx \leq c_3\|f\|_{PW_+^{2,\nu}} e^{2|y|},$$

and for any $z = x+iy \in \mathbb{C}$ holds

$$|f(z)| \leq c_4\|f\|_{PW_+^{2,\nu}} e^{|y|}(1+|z|)^{-\nu-1/2}(1+|y|)^{-1/2}.$$

Lemma 3.3. (see [17, pp. 362, 363, 366, 367]) *Let $\nu \geq -1/2$. If the sequence $\mathrm{P} = \{\rho_k : k \in \mathbb{N}\}$ is a complete interpolating sequence for $PW_+^{2,\nu}$, then for every $k \in \mathbb{Z}\setminus\{0\}$ the interpolation problems $f_k(\lambda_k) = f_k(\lambda_{-k}) = 1$ and $f_k(\lambda_j) = f_k(\lambda_{-j}) = 0$, $j \in \mathbb{Z}\setminus\{0;k;-k\}$, are solvable in $H^2(\mathbb{C}_{0,+\infty})$ and in $H^2(\mathbb{C}_{-\infty,0})$. Moreover, properties 1), 2) and the following ones hold:*

$$\inf\left\{\prod_{\substack{\operatorname{Im}\lambda_j>a,\\ j\neq k}} \left|\frac{\lambda_k-\lambda_j}{\lambda_k-\overline{\lambda}_j+2ia}\right| : \operatorname{Im}\lambda_k > a\right\} > 0 \quad \textit{for each } a\in\mathbb{R}; \tag{3.1}$$

$$\inf\left\{\prod_{\substack{\operatorname{Im}\lambda_j<a,\\ j\neq k}} \left|\frac{\lambda_k-\lambda_j}{\lambda_k-\overline{\lambda}_j-2ia}\right| : \operatorname{Im}\lambda_k < a\right\} > 0 \quad \textit{for each } a\in\mathbb{R}; \tag{3.2}$$

$$\sup\left\{\sum_{\substack{j\in\mathbb{Z}\setminus\{0\},\\ j\neq k}} \frac{(1+|\operatorname{Im}\lambda_k|)(1+|\operatorname{Im}\lambda_j|)}{|\lambda_k-\overline{\lambda}_j|^2} : k\in\mathbb{Z}\setminus\{0\}\right\} < +\infty; \tag{3.3}$$

for some $\varepsilon > 0$ the disks

$$K(\lambda_k) := \{z : |z-\lambda_k| \leq 10\varepsilon(1+|\operatorname{Im}\lambda_k|)\} \quad \textit{are pairwise disjoint;} \tag{3.4}$$

$$\mu_\Lambda := \sum_{\operatorname{Im}\lambda_k\geq 0} \operatorname{Im}\lambda_k \delta_{\lambda_k} \tag{3.5}$$

is a Carleson measure in $\mathbb{C}^+ := \mathbb{C}_{0,+\infty}$ (δ_λ is the unit point measure at λ).

By manipulating the Carleson conditions (3.1) and (3.2) in much the same way as in [7, pp. 288–290], we obtain the following lemma.

Lemma 3.4. (see [17, pp. 363, 364, 367]) *Condition* (3.3) *is equivalent to conditions 3) and 4), and also to conditions* (3.1) *and* (3.2).

Lemma 3.5. *Let $\nu \geq -1/2$. If a sequence Λ satisfies condition* (3.3), *then*

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} (1+|\operatorname{Im}\lambda_k|)(1+|\lambda_k|)^{2\nu+1}e^{-2|\operatorname{Im}\lambda_k|}|f(\lambda_k)|^2 \leq c_5\|f\|^2_{PW_+^{2,\nu}},\ f\in PW_+^{2,\nu}.$$

Proof. Indeed, if $f \in PW_+^{2,\nu}$, then by Lemma 3.1, we have $f_+(z) = (z+2i)^{\nu+1/2} e^{iz} f(z) \in H^2(\mathbb{C}_{-2,+\infty})$ and $f_-(z) = (z-2i)^{\nu+1/2} e^{-iz} f(z) \in H^2(\mathbb{C}_{-\infty,2})$, and for the Hardy spaces the corresponding result is true (see [15]). □

Let $Q(x;r)$ be the square with center at $x \in \mathbb{R}$, side length $2r$, and sides parallel to the coordinate axes. According to [17], we say that a set $\mathrm{P} \subset \mathbb{C}$ is *relatively dense* if there exists $r_0 > 0$ such that $\mathrm{P} \cap Q(x;r_0) \neq \varnothing$ for each $x \in \mathbb{R}$.

Lemma 3.6. *Let $\nu \geq -1/2$. If $\mathrm{P} = \{\rho_k : k \in \mathbb{N}\}$ is a complete interpolating sequence for $PW_+^{2,\nu}$, then $\Lambda = \{\lambda_k : k \in \mathbb{Z} \setminus \{0\}\}$ is relatively dense and for any function $f \in PW_+^{2,\nu}$*

$$\|f\|^2_{PW_+^{2,\nu}}/c_6 \leq \sum_{k \in \mathbb{Z}\setminus\{0\}} (1 + |\operatorname{Im}\lambda_k|)(1+|\lambda_k|)^{2\nu+1} e^{-2|\operatorname{Im}\lambda_k|} |f(\lambda_k)|^2 \leq c_6 \|f\|^2_{PW_+^{2,\nu}}. \tag{3.6}$$

Proof. The proof of condition (3.6) is the same as that of [17]. Assume that a set Λ is not relatively dense. Then there exist sequences $\{x_j\}$ and $\{r_j\}$ such that $|x_j| > s$, $s \in \mathbb{N}$, $r_j \to +\infty$ and $\Lambda \cap Q(x_j;r_j) = \varnothing$. Let $s > 1+\nu$, $\mu = \nu + 1 - s$, $q_j = 1/|x_j|$ and let $f_j(z) = q_j^\mu \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} q_j^{2k+1} (z^2 - x_j^2)^k \right)^s$. Then $f_j \in PW_+^{2,\nu}$,

$$f_j(z) = q_j^\mu \left(\frac{\sin q_j (z^2 - x_j^2)^{1/2}}{(z^2 - x_j^2)^{1/2}} \right)^s,$$

$$\|f_j\|^2_{PW_+^{2,\nu}} = 2\int_0^1 (1-u^2)^\nu \left(\frac{|\sin u|}{u}\right)^{2s} u\,du + 2\int_0^{+\infty} (1+u^2)^\nu \left(\frac{|\sin u|}{u}\right)^{2s} u\,du,$$

$$\|f_j\|^2_{PW_+^{2,\nu}}/c_6 \leq \sum_{k=1}^{\infty} (1 + |\operatorname{Im}\lambda_k|)(1+|\lambda_k|)^{2\nu+1} e^{-2|\operatorname{Im}\lambda_k|} |f_j(\lambda_k)|^2 \leq c_6 \|f_j\|^2_{PW_+^{2,\nu}},$$

and

$$\|f_j\|^2_{PW_+^{2,\nu}}/c_6 \leq \sum_{k=1}^{\infty} (1 + |\operatorname{Im}\lambda_k|)(1+|\lambda_k|)^{2\nu+1} e^{-2|\operatorname{Im}\lambda_k|} |f_j(\lambda_k)|^2 \to 0, \quad j \to \infty.$$

We have a contradiction, since $\|f_j\|^2_{PW_+^{2,\nu}}$ is independent of j. □

Lemma 3.7. *Let $\nu \geq -1/2$. If $\mathrm{P} = \{\rho_k : k \in \mathbb{N}\}$ is a complete interpolating sequence for $PW_+^{2,\nu}$, then there exists a relatively dense set $\Gamma = \{\gamma_j : j \in \mathbb{Z}\setminus\{0\}\} \subset \Lambda$ such that $(|\gamma_j|^{2\nu+1}|S'(\gamma_j)|^2)$ satisfies the discrete (A_2) condition.*

Proof. Indeed, let $r > r_0$, $Q_j = Q(4jr;r)$ and $\Gamma = \{\gamma_j : j \in \mathbb{Z}\setminus\{0\}\} \subset \Lambda$ is a relatively dense sequence such that $\gamma_j \in Q_j$. Let $\Sigma = \{\sigma_j\}$ be another sequence with $|\gamma_j - \sigma_j| = \varepsilon$ and $S(\sigma_j) = \varepsilon S'(\gamma_j)$. Then the sequence $\{\sigma_j\}$ also satisfies condition (3.3). Therefore, by Lemma 3.5, we have

$$\sum_j |\sigma_j|^{2\nu+1} (1 + |\operatorname{Im}\sigma_j|) e^{-2|\operatorname{Im}\sigma_j|} |f(\sigma_j)|^2 \leq c_7 \|f\|^2_{PW_+^{2,\nu}}.$$

Since S' is an odd function, then for a finite set $\{d_j : j \in [1;m] \cap \mathbb{Z}\}$ the unique solution of the interpolation problem $f(\gamma_k) = d_k$, $\gamma_k \in \Gamma$ and $f(\gamma_k) = 0$, $\gamma_k \notin \Gamma$, has the form

$$f(z) = \sum_{k=1}^{m} \frac{2\gamma_k d_k S(z)}{(z^2 - \gamma_k^2) S'(\gamma_k)} = \sum_{k=-m, k\neq 0}^{m} \frac{d_k S(z)}{(z-\gamma_k) S'(\gamma_k)}, \quad d_{-k} := d_k.$$

In this case, according to Lemma 3.6, we get

$$c_6\|f\|^2_{PW^{2,\nu}_+} \le \sum_j |\gamma_j|^{2\nu+1}(1+|\operatorname{Im}\gamma_j|)e^{-2|\operatorname{Im}\gamma_j|}|d_j|^2.$$

Since $\gamma_j \in Q_j$, then the sequences $(|\operatorname{Im}\gamma_j|)$ and $(|\operatorname{Im}\sigma_j|)$ are bounded. Therefore

$$\sum_j |\sigma_j|^{2\nu+1}(1+|\operatorname{Im}\sigma_j|)e^{-2|\operatorname{Im}\sigma_j|}|f(\sigma_j)|^2 \le c_8 \sum_j |\gamma_j|^{2\nu+1}(1+|\operatorname{Im}\gamma_j|)e^{-2|\operatorname{Im}\gamma_j|}|d_j|^2$$

and

$$\sum_j |\sigma_j|^{2\nu+1}|f(\sigma_j)|^2 \le c_9 \sum_j |\gamma_j|^{2\nu+1}|d_j|^2.$$

Since

$$f(\sigma_j) = \varepsilon S'(\gamma_j)\sum_k \frac{d_k}{(\sigma_j-\gamma_k)S'(\gamma_k)},$$

we obtain

$$\sum_j |\gamma_j|^{2\nu+1}|S'(\gamma_j)|^2\left|\sum_k \frac{d_k}{\sigma_j-\gamma_k}\right|^2 \le c_9\sum_j |\gamma_j|^{2\nu+1}|S'(\gamma_j)|^2|d_j|^2.$$

Thus, the operator $\mathcal{H}_{\Gamma,\Sigma} : d = \{d_j\} \longmapsto \mathcal{H}_{\Gamma,\Sigma}d$, $\mathcal{H}_{\Gamma,\Sigma}d := \sum\limits_k \frac{d_k}{\sigma_j-\gamma_k}$ is bounded on $l^{2,w}_+$ if $w_j = |\gamma_j|^{2\nu+1}|S'(\gamma_j)|^2$. Therefore ([17]), the sequence $(|\gamma_j|^{2\nu+1}|S'(\gamma_j)|^2)$ satisfies the discrete (A_2) condition, and the lemma is proved. □

Lemma 3.8. *If $\nu \ge -1/2$ and $\mathrm{P} = \{\rho_k : k \in \mathbb{N}\}$ is a complete interpolating sequence for $PW^{2,\nu}_+$, then 5) holds.*

In fact, since the discrete (A_2) condition is equivalent to the continuous (A_2) condition, by using Lemma 2 from [17, p. 368] and the inequality $t^s s^{1-\alpha} \le t+s$, t, $s>0$, $\alpha \in [0;1]$, we obtain that the statement of this lemma follows from Lemma 3.7.

Similarly to [17], by using conditions 1), 2) and 5), we obtain the following lemma.

Lemma 3.9. (see [17, Lemma 3, p. 371]) *Let condition* (3.4) *be true. Then*

$$|S(z)| \ge c_{10}(1+|z|)^{-1/2}e^{|\operatorname{Im} z|}, \quad \text{for} \quad \operatorname{dist}(z;\Lambda) \ge \varepsilon(1+|\operatorname{Im} z|).$$

Lemma 3.10. *Let $\nu \ge -1/2$ and $\mathrm{P} = \{\rho_k : k \in \mathbb{N}\}$ be an arbitrary sequence of nonzero complex numbers. A sequence $\Lambda = \{\lambda_k : k \in \mathbb{Z}\setminus\{0\}\}$ is a complete interpolating sequence for $PW^{2,\nu}_+$ if and only if conditions 1) – 5) hold.*

Proof. The *necessity* follows from Lemmas 3.3–3.8. Let us prove the *sufficiency*. First, we will show that the function

$$f(z) = \sum_{m\in\mathbb{N}} \frac{2\rho_m d_m S(z)}{(z^2-\rho_m^2)S'(\lambda_m)} = v.p.\sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{d_m S(z)}{(z-\lambda_m)S'(\lambda_m)} \tag{3.7}$$

is the required solution of the interpolation problem $f(\rho_k) = d_k$, where $d_{-k} := d_k$ for $k \in \mathbb{N}$. To this end, it suffices to estimate the partial sums of series (3.7) corresponding to $\lambda_m \in \mathbb{R}\cup\mathbb{C}_{0,+\infty}$ and $\lambda_m \in \mathbb{R}\cup\mathbb{C}_{-\infty,0}$, respectively on the lines $\operatorname{Im} z = -1/2$ and $\operatorname{Im} z = 1/2$. Following [17, p. 373], we will give the corresponding estimates on the line $\operatorname{Im} z = 0$ assuming that $\operatorname{Im}\lambda_m \geq 1/2$ and $\operatorname{Im}\lambda_m < 1/2$, respectively. In the first case, let

$$B(z) = \prod_{\operatorname{Im}\lambda_j \geq 1/2} \frac{z-\lambda_j}{z-\overline{\lambda}_j}, \quad G(z) = \frac{S(z)}{e^{-iz}B(z)}.$$

Then $S(z) = G(z)e^{-iz}B(z)$, where G is a bounded outer function in $\mathbb{C}^+$, and we observe that 5) is equivalent to $|G(x)|^2$ satisfying the (A_2) condition. Moreover, $|G(x)|^{-2}$ satisfies the (A_2) condition, and the Lemma 3.4 implies $|S'(\lambda_k)| \asymp |G(\lambda_k)|\frac{e^{\operatorname{Im}\lambda_k}}{\operatorname{Im}\lambda_k}$. Now let

$$\mathcal{H} : f \longmapsto \mathcal{H}f(t) = \frac{1}{i\pi}\int_{-\infty}^{+\infty} \frac{f(\tau)}{t-\tau}\, d\tau$$

is the classical Hilbert operator. Following [17, p. 373], we consider the function

$$\widetilde{f}(x) = \sum_{\substack{\operatorname{Im}\lambda_m \geq 1/2,\\ m=-n, m\neq 0}}^{n} \frac{\operatorname{Im}\lambda_m d_m e^{-\operatorname{Im}\lambda_m} G(x)}{(x-\lambda_m)G(\lambda_m)}.$$

By duality (see [17, p. 373]), we obtain (here $w(x) = x^{\nu+1/2}$, $h_w(x) = w(x)h(x) \in L^2(\mathbb{R})$)

$$\begin{aligned}
\|\widetilde{f}\|_{L^{2,\nu}(\mathbb{R})} &= \sup_{\substack{\|h\|\leq 1,\\ h\in L^{2,\nu}(\mathbb{R})}} \left| \sum_{\substack{\operatorname{Im}\lambda_m \geq 1/2,\\ m=-n, m\neq 0}}^{n} \int_{-\infty}^{+\infty} \frac{\operatorname{Im}\lambda_m d_m e^{-\operatorname{Im}\lambda_m} G(x)h_w(x)}{(x-\lambda_m)G(\lambda_m)}\, dx \right| \\
&\leq \sup_{\substack{\|h\|\leq 1,\\ h\in L^{2,\nu}(\mathbb{R})}} \left| \sum_{\substack{\operatorname{Im}\lambda_m \geq 1/2,\\ m=-n, m\neq 0}}^{n} \frac{\operatorname{Im}\lambda_m d_m e^{-\operatorname{Im}\lambda_m} \mathcal{H}Gh_w(\lambda_m)}{G(\lambda_m)} \right| \\
&\leq \sup_{\substack{\|h\|\leq 1,\\ h\in L^{2,\nu}(\mathbb{R})}} \left(\sum_{\operatorname{Im}\lambda_m \geq 1/2} \left| \frac{\mathcal{H}Gh_w}{G}(\lambda_m) \right|^2 \operatorname{Im}\lambda_m \right)^{1/2} \\
&\quad \times \left(\sum_{\operatorname{Im}\lambda_m \geq 1/2} \operatorname{Im}\lambda_m e^{-2\operatorname{Im}\lambda_m} |d_m|^2 \right)^{1/2}.
\end{aligned}$$

Since $\sum_{\operatorname{Im}\lambda_k \geq 0} \operatorname{Im}\lambda_k \delta_{\lambda_k}$ is a Carleson measure, $|G(x)|^{-2}$ satisfies the (A_2) condition, G is an outer function in $\mathbb{C}^+$, we have $\mathcal{H}Gh/G \in H^2(\mathbb{C}^+)$. Therefore, the last sums are uniformly bounded, and we get the desired conclusion. The sum corresponding to $\operatorname{Im}\lambda_k < 0$ is treated similarly. Hence, there exists a solution of the considered interpolation problem. Now we turn to the proof of uniqueness. Observe first that (see [17, p. 370])

$$\int_{-\infty}^{+\infty} |F(x)|^2 \frac{dx}{1+|x|^2} < +\infty, \quad \int_{-\infty}^{+\infty} |F(x)|^2\, dx = +\infty.$$

Suppose that $f(\rho) = 0$, $\rho \in \mathrm{P}$. Let $\psi(z) = f(z)/S(z)$. Since, by Lemma 3.2, $|f(z)| \leq c_4 \|f\|_{PW_+^{2,\nu}} e^{|\operatorname{Im} z|}(1+|z|)^{-\nu-1/2}(1+|\operatorname{Im} z|)^{-1/2}$, $z \in \mathbb{C}$, if $f \in PW_+^{2,\nu}$, then using Lemma 3.9, we obtain

$$|\psi(z)| = \left|\frac{f(z)}{S(z)}\right| \leq c_{11} \frac{e^{|\operatorname{Im} z|}(1+|z|)^{-\nu-1/2}(1+|\operatorname{Im} z|)^{-1/2}}{(1+|z|)^{-1/2}e^{|\operatorname{Im} z|}} = c_{11}\frac{(1+|z|)^{-\nu}}{(1+|\operatorname{Im} z|)^{1/2}}.$$

Therefore, $|\psi(z)|$ is uniformly bounded for z satisfying $\operatorname{dist}(z;\Lambda) \geq \varepsilon(1+|\operatorname{Im} z|)$. By the classical Phragmén-Lindelöf principle ([16, p. 39]), we get $\psi(z) \equiv c_{12}$, whereas ([17, p. 372]) $\int_{-\infty}^{+\infty} |S(x+i)|^2\,dx = +\infty$ and $\int_{-\infty}^{+\infty} |S(x)|^2\,dx = +\infty$. Hence, $\psi(z) \equiv 0$. □

Theorem 1.1 is an immediate corollary of Lemma 3.10 and Theorem 1.2.

Acknowledgments

The authors thank the unknown Referee for a careful reading of this paper. The first version of the paper was significantly improved by the suggestions of the Referee.

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Received: 27.03.2019