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COMPLETENESS OF THE SYSTEMS OF BESSEL FUNCTIONS OF NEGATIVE HALF–INTEGER INDEX LESS THAN -1 IN WEIGHTED L^2–SPACES

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Abstract. We establish necessary and sufficient conditions for the completeness of the system $\{\rho_k^m\sqrt{x\rho_k}J_{-m-1/2}(x\rho_k) : k\in\mathbb{N}\}$ in the space $L^2((0;1);x^{2m}dx)$ in terms of sequences of zeros of functions from certain classes of entire functions, where $L^2((0;1);t^{2m}dt)$ be the weighted Lebesgue space of all measurable functions $f:(0;1)\to\mathbb{C}$, satisfying $\int_0^1 t^{2m}|f(t)|^2\,dt<+\infty$, $m\in\mathbb{N}$, $J_{-m-1/2}$ be the Bessel function of the first kind of index $-m-1/2$ and $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of distinct nonzero complex numbers. We also obtain an analog of the Paley-Wiener theorem related to this system.

1. Introduction

Let (see, for example, [3, p. 4], [25, p. 345], [35, p. 40])

$$J_\nu(z)=\sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)},\quad z=x+iy=re^{i\varphi},$$

be the Bessel function of the first kind of index $\nu\in\mathbb{R}$, where Γ is the classical gamma function. By Hurwitz's theorem (see [3, p. 59], [35, p. 483]), for $\nu>1$ the function $J_{-\nu}$ has an infinity of real zeros and also $2[\nu]$ pairwise conjugate complex zeros, among them two pure imaginary zeros when $[\nu]$ is an odd integer (here $[\nu]$ is the integer part of ν). Let ρ_k, $k\in\mathbb{N}$, be the zeros of the function $J_{-\nu}$ for which $\Im\rho_k>0$ if $\rho_k\in\mathbb{C}$ or $\rho_k>0$ if $\rho_k\in\mathbb{R}$.

Let $L^2(X)$ be the space of all measurable functions $f:X\to\mathbb{C}$ on a measurable set $X\subseteq\mathbb{R}$ with the norm

$$\|f\|_{L^2(X)}=\left(\int_X|f(x)|^2\,dx\right)^{1/2},$$

let $\gamma\in\mathbb{R}$ and $L^2((0;1);t^\gamma dt)$ be the weighted Lebesgue space of all measurable functions $f:(0;1)\to\mathbb{C}$, satisfying

$$\int_0^1 t^\gamma|f(t)|^2\,dt<+\infty.$$

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The function h belongs to the space $L^2((0;1);t^\gamma dt)$ if and only if the function $q(t) = t^{\gamma/2}h(t)$ belongs to $L^2(0;1)$. Since (see [25, p. 350], [35, p. 55])

$$J_{-m-1/2}(z) = \sqrt{\frac{2}{\pi}} z^{m+1/2} \left(\frac{d}{zdz}\right)^m \left(\frac{\cos z}{z}\right), \quad m \in \mathbb{N},$$

the function $f(t) = \rho^m \sqrt{t\rho} J_{-m-1/2}(t\rho)$ belongs to the space $L^2((0;1);t^{2m}dt)$ for every $m \in \mathbb{N}$ and $\rho \in \mathbb{C}$. A system of elements $\{u_k : k \in \mathbb{N}\}$ in a separable Hilbert space $\mathscr{H}$ is called complete (see, e.g., [17, p. 131], [18, p. 4258]) if $\overline{\operatorname{span}}\{u_k : k \in \mathbb{N}\} = \mathscr{H}$.

It is known that the approximation properties of the system $\{\sqrt{x}J_\nu(x\widetilde{\rho}_k) : k \in \mathbb{N}\}$ with $\nu > -1$ depends on the properties of the sequence $(\widetilde{\rho}_k)_{k\in\mathbb{N}}$. The classical results relate mainly to the case when $(\widetilde{\rho}_k)_{k\in\mathbb{N}}$ is a sequence of positive zeros of J_ν (see, for instance, [1, 3, 4, 6, 7, 21, 25, 35]). In particular, it is well known that the system $\{\sqrt{x}J_\nu(x\widetilde{\rho}_k) : k \in \mathbb{N}\}$ is an orthogonal basis for the space $L^2(0;1)$ if $\nu > -1$ and $(\widetilde{\rho}_k)_{k\in\mathbb{N}}$ is a sequence of positive zeros of J_ν (see [1, 3, 4, 6, 7, 21, 25, 35]). From this it follows that (see [6, 7, 25, 35]) if $\nu > -1$ and $(\widetilde{\rho}_k)_{k\in\mathbb{N}}$ is a sequence of positive zeros of J_ν, then the system $\{x^{-\nu}J_\nu(x\widetilde{\rho}_k) : k \in \mathbb{N}\}$ is complete and minimal in $L^2((0;1);x^{2\nu+1}dx)$. The system $\{\sqrt{x}J_\nu(x\widetilde{\rho}_k) : k \in \mathbb{N}\}$ with $\nu > -1$ is also complete (see [25, pp. 347, 356], [6, 35]) in $L^2(0;1)$ if $\widetilde{\rho}_k J'_\nu(\widetilde{\rho}_k) + \lambda J_\nu(\widetilde{\rho}_k) = 0$ where $\lambda + \nu > 0$. In addition, from [4] it follows that if $\nu > -1/2$ and $(\widetilde{\rho}_k)_{k\in\mathbb{N}}$ is a sequence of distinct positive numbers such that $\widetilde{\rho}_k \leqslant \pi(k + \nu/2)$ for all sufficiently large $k \in \mathbb{N}$, then the system $\{\sqrt{x}J_\nu(x\widetilde{\rho}_k) : k \in \mathbb{N}\}$ is complete in $L^2(0;1)$.

Basis properties (completeness, minimality, basicity) of a system $\{\sqrt{x\widetilde{\rho}_k}J_\nu(x\widetilde{\rho}_k) : k \in \mathbb{N}\}$ in $L^2(0;1)$ and more general systems $\{x^{-p-1}\sqrt{x\widetilde{\rho}_k}J_\nu(x\widetilde{\rho}_k) : k \in \mathbb{N}\}$ in the weighted spaces $L^2((0;1);x^{2p}dx)$, where $\nu \geqslant 1/2$, $p \in \mathbb{R}$ and $(\widetilde{\rho}_k)_{k\in\mathbb{N}}$ is a sequence of distinct nonzero complex numbers, have been studied in [8, 9, 28, 29, 30, 31, 32]. In these papers, criteria for completeness, minimality and basicity of the above systems of Bessel functions were found in terms of sequences of zeros of functions from certain classes of entire functions.

At the same time, the approximation properties of the systems of Bessel functions for $\nu < -1$, $\nu \notin \mathbb{Z}$, were investigated in [11, 12, 13, 15, 19, 20, 26, 27, 33, 34]. Such systems of the Bessel functions appear in the study of some non-classical boundary-value problems (see [11, 12, 20, 33, 34]), whose singularity lies in the fact that the set of their canonical eigenfunctions can be over-complete, which means that it remains complete after elimination of a certain number of these eigenfunctions. In particular, at studying of one boundary-value problem for the Bessel equation, in [33] (see also [34]) it was proven that the system $\{\rho_k\sqrt{x\rho_k}J_{-3/2}(x\rho_k) : k \in \mathbb{N}\}$ is complete in the space $L^2((0;1);x^2dx)$, and the system $\{\rho_k\sqrt{x\rho_k}J_{-3/2}(x\rho_k) : k \in \mathbb{N}\setminus\{1\}\}$ is complete, minimal and is not a basis in this space, where $(\rho_k)_{k\in\mathbb{Z}\setminus\{0\}}$, $\rho_{-k} := -\rho_k$, is a sequence of zeros of the function $J_{-3/2}$. In addition, in [27] it was shown that the biorthogonal system $\{\pi(1+\rho_k^2)\rho_k^{-4}x^{-2}(\rho_k\sqrt{x\rho_k}J_{-3/2}(x\rho_k) - \rho_1\sqrt{x\rho_1}J_{-3/2}(x\rho_1)) : k \in \mathbb{N}\setminus\{1\}\}$ is also complete in $L^2((0;1);x^2dx)$. Further, in [19] it was proven that the system $\{\rho_k^2\sqrt{x\rho_k}J_{-5/2}(x\rho_k) : k \in \mathbb{N}\setminus\{1;2\}\}$ is complete and minimal in $L^2((0;1);x^4dx)$, where $(\rho_k)_{k\in\mathbb{N}}$ is a sequence of zeros of $J_{-5/2}$. Furthermore, in [20] it was established that the system $\left\{\rho_k^{\nu-1/2}\sqrt{x\rho_k}J_{-\nu}(x\rho_k) : k \in \mathbb{N}\setminus\{1;2;\ldots;m\}\right\}$ is complete in $L^2((0;1);x^{2\nu-1}dx)$ if $\nu = m+1/2$, $m \in \mathbb{N}$ and $(\rho_k)_{k\in\mathbb{N}}$ is a sequence of zeros of $J_{-\nu}$.

Approximation properties of the systems of Bessel functions of negative half-integer index less than -1 are of significant interest in the study of some boundary-value problems for the Bessel equation [19, 20, 26, 27, 33, 34] and generalized eigenvectors for certain Bessel-type differential operators [11, 12]. An important role in the study of approximation properties of the systems of Bessel functions of this index is also played by the case when $(\rho_k)_{k\in\mathbb{N}}$ is an arbitrary sequence of distinct nonzero complex numbers. In this direction, in [13, 26] the authors obtained a criteria for the completeness and minimality of the system $\left\{\rho_k\sqrt{x\rho_k}J_{-3/2}(x\rho_k) : k\in\mathbb{N}\right\}$ in $L^2((0;1);x^2dx)$ with an arbitrary sequence of nonzero complex numbers $(\rho_k)_{k\in\mathbb{N}}$. In addition, in [27] it was proven that the system $\left\{x^{-2}(\rho_1^2-\rho_k^2)^{-1}(\rho_k\sqrt{x\rho_k}J_{-3/2}(x\rho_k)-\rho_1\sqrt{x\rho_1}J_{-3/2}(x\rho_1)) : k\in\mathbb{N}\setminus\{1\}\right\}$ is also complete and minimal in $L^2((0;1);x^2dx)$, where $(\rho_k)_{k\in\mathbb{N}}$ is a sequence of distinct nonzero complex numbers such that $\rho_k^2\neq\rho_n^2$ for $k\neq n$. Besides, in [15] were obtained some necessary and sufficient conditions for the completeness of the system $\left\{\rho_k^2\sqrt{x\rho_k}J_{-5/2}(x\rho_k) : k\in\mathbb{N}\right\}$ in the space $L^2((0;1);x^4dx)$ in terms of an entire function with the set of zeros coinciding with the sequence of distinct nonzero complex numbers $(\rho_k)_{k\in\mathbb{N}}$. It is an actual problem to perform similar investigations for the completeness of more general systems $\left\{\rho_k^{\nu-1/2}\sqrt{x\rho_k}J_{-\nu}(x\rho_k) : k\in\mathbb{N}\right\}$ in $L^2((0;1);x^{2\nu-1}dx)$ in the case where $\nu\notin\mathbb{Z}$ and $\nu>1$.

The aim of this paper is to establish some necessary and sufficient conditions for the completeness of the system $\left\{\psi_{k,m} : k\in\mathbb{N}\right\}$ with $\psi_{k,m}(x) := \rho_k^m\sqrt{x\rho_k}J_{-m-1/2}(x\rho_k)$ and $m\in\mathbb{N}$ in the space $L^2((0;1);x^{2m}dx)$, where $(\rho_k)_{k\in\mathbb{N}}$ is an arbitrary sequence of distinct nonzero complex numbers (see Theorems 2–8). We also obtain an analogue of the Paley-Wiener theorem related to this system (see Theorem 1). This generalizes the results of papers [10, 13, 14, 15, 19, 20, 26, 27, 33, 34].

2. Preliminaries

An entire function G is said to be of exponential type $\sigma\in[0;+\infty)$ (see [17, p. 4], [18, p. 4262]) if for any $\varepsilon>0$ there exists a constant $c(\varepsilon)$ such that

$$|G(z)| \leqslant c(\varepsilon)\exp((\sigma+\varepsilon)|z|), \quad z\in\mathbb{C}.$$

Denote by PW_σ^2 the set of all entire functions of exponential type $\sigma\in(0;+\infty)$ whose narrowing on $\mathbb{R}$ belongs to the space $L^2(\mathbb{R})$, and by $PW_{\sigma,+}^2$ denote the class of even entire functions from PW_σ^2. According to the Paley-Wiener theorem (see [17, p. 69], [18, p. 4263]), the class PW_σ^2 coincides with the class of functions G admitting the representation

$$G(z) = \int_{-\sigma}^{\sigma} e^{itz}g(t)\,dt, \quad g\in L^2(-\sigma;\sigma),$$

and the class $PW_{\sigma,+}^2$ consists of the functions G representable in the form

$$G(z) = \int_0^{\sigma}\cos(tz)g(t)\,dt, \quad g\in L^2(0;\sigma).$$

Moreover, $\|g\|_{L^2(0;\sigma)} = \sqrt{2/\pi}\|G\|_{L^2(0;+\infty)}$ and

$$g(t) = \frac{2}{\pi}\int_0^{+\infty} G(z)\cos(tz)\,dz.$$

The Hankel transform of order $\nu \geqslant -1/2$ of a function $f \in L^2(0;+\infty)$, is defined by (see [16, 21, 22, 24, 35])

$$f(z) = \int_0^{+\infty} \sqrt{tz}J_\nu(tz)g(t)\,dt, \quad g \in L^2(0;+\infty),$$

if the integral converges in some sense (absolutely, improper, or mean convergence). Analogues of the Paley-Wiener theorem for the Hankel transform of order $\nu \geqslant -1/2$ were established in [2, 5, 21, 22, 23, 24, 36]. In addition, in [16] were obtained some analogues of the Paley-Wiener theorem for the Hankel-type transform of half-integer order less than -1 of even entire functions Q of exponential type $\sigma \leqslant 1$ of the kind

$$Q(z) = z^m \int_0^1 \sqrt{tz}J_{-m-1/2}(tz)t^{2m}h(t)\,dt, \quad h \in L^2((0;1);x^{2m}dx), \quad m \in \mathbb{N}. \tag{1}$$

These Paley-Wiener-type theorems give a description of the class of even entire functions Q of exponential type $\sigma \leqslant 1$ under this transformation in terms of the existence of certain solutions of some differential equations (see [16]). Similar results in special cases $m \in \{1;2;3\}$, were obtained in [10, 14, 26].

EXAMPLE 1. ([13]) The function

$$Q(z) = -\sqrt{\frac{2}{\pi}}\frac{\cos z}{z^2 - \pi^2/4}\left(1 - \frac{4(-\pi+2)}{\pi^3}(z^2 - \pi^2/4)\right)$$

can be represented in the form (1) with $m = 1$ and

$$h(t) = \frac{4}{t^3\pi^3}\left(2 - 2\cos\left(\frac{\pi}{2}t\right) - \pi t\sin\left(\frac{\pi}{2}t\right)\right).$$

The function $Q(z) = \cos z$ cannot be represented in the form (1) with $m = 1$.

EXAMPLE 2. ([10]) The function

$$Q(z) = \sqrt{\frac{2}{\pi}}z\frac{\left(1 - \frac{4(-\pi+2)}{\pi^3}(z^2 - \pi^2/4)\right)(z^2 - \pi^2/4)\sin z + 2z\cos z}{(z^2 - \pi^2/4)^2} + 3\sqrt{\frac{2}{\pi}}\frac{\cos z}{z^2 - \pi^2/4}\left(1 - \frac{4(-\pi+2)}{\pi^3}(z^2 - \pi^2/4)\right)$$

admits the representation (1) with $m = 2$ and

$$h(t) = \frac{4}{t^4\pi^3}\left(2 - 2\cos\left(\frac{\pi}{2}t\right) - \pi t\sin\left(\frac{\pi}{2}t\right)\right).$$

The function $Q(z) = 2\sqrt{2/\pi}z^4\cos z$ does not admits the representation (1) with $m = 2$.

EXAMPLE 3. ([14, 16]) The function

$$Q(z) = -4\sqrt{\frac{2}{\pi}}\frac{z^2}{(z^2-\pi^2/4)^3}\left(2z^2\cos z + (z^2-\pi^2/4)(z\sin z + 3\cos z)\right) + \sqrt{\frac{2}{\pi}}\frac{z^2\cos z - 7z\sin z - 15\cos z}{z^2-\pi^2/4}\left(1 - \frac{4(-\pi+2)}{\pi^3}(z^2-\pi^2/4)\right)$$

can be represented in the form (1) with $m = 3$ and

$$h(t) = \frac{4}{t^5\pi^3}\left(2 - 2\cos\left(\frac{\pi}{2}t\right) - \pi t\sin\left(\frac{\pi}{2}t\right)\right).$$

The function $Q(z) = \sqrt{2/\pi}z^8\cos z$ cannot be represented in the form (1) with $m = 4$.

Let $\log^+ x = \max(0; \log x)$ for $x > 0$. Here and subsequently, by c_1, c_2, ... we denote arbitrary positive constants.

To prove our main results we need the following auxiliary lemmas.

LEMMA 1. ([16]) *Let an entire function Q be defined by the formula* (1)*. Then Q is an even entire function of exponential type $\sigma \leqslant 1$ such that for all $z := x+iy \in \mathbb{C}$ and $m \in \mathbb{N}$, we have*

$$|Q(z)| \leqslant c_1 \frac{e^{|\Im z|}}{\sqrt{1+|\Im z|}}(1+|z|)^m.$$

LEMMA 2. (see [17, p. 127], [18, p. 4263]) *Let Q be an entire function of exponential type $\sigma \leqslant 1$ such that*

$$\int_{-\infty}^{+\infty} \frac{\log^+|Q(x)|}{1+x^2}\,dx < +\infty,$$

and let $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of nonzero roots of the function $Q(z)$. Then

$$\sum_{k\in\mathbb{N}}\left|\Im\frac{1}{\rho_k}\right| < +\infty.$$

3. Main results

Let $\left(z^{-1}d/dz\right)^m$ be the m-th power of the differential operator $z^{-1}d/dz$. Our principal results are the following statements.

THEOREM 1. *Let $m \in \mathbb{N}$. An entire function Q has the representation* (1) *with some function $h \in L^2((0;1); x^{2m}dx)$ if and only if it is an even entire function of exponential type $\sigma \leqslant 1$ such that*

$$Q(0) = \frac{2^{m+1/2}}{\Gamma(-m+1/2)}\int_0^1 t^m h(t)\,dt, \tag{2}$$

$$\left(\frac{1}{z}\frac{d}{dz}\right)^{l} Q(z)\bigg|_{z=0} = \frac{(-1)^l 2^{m-l+1/2}}{\Gamma(-m+l+1/2)}\int_0^1 t^{2l+m}h(t)\,dt, \quad l\in\{1;2;\ldots;m-1\}, \quad (3)$$

and the function $\left(z^{-1}d/dz\right)^m Q(z)$ *belongs to the space* $PW^2_{1,+}$. *If these conditions are fulfilled, then*

$$h(t) = \sqrt{\frac{2}{\pi}}(-1)^m \frac{1}{t^{3m}}\int_0^{+\infty}\cos(tz)\left(\frac{1}{z}\frac{d}{dz}\right)^m Q(z)\,dz.$$

Proof. Necessity. Let $m\in\mathbb{N}$ and Q has the representation (1) with some function $h\in L^2((0;1);x^{2m}dx)$. Then

$$Q(z) = \int_0^1 (tz)^{m+1/2}J_{-m-1/2}(tz)t^m h(t)\,dt.$$

Since

$$z^{\nu}J_{-\nu}(z) = \sum_{k=0}^{\infty}\frac{(-1)^k 2^{\nu-2k}z^{2k}}{k!\Gamma(-\nu+k+1)}, \quad \nu\in\mathbb{R}, \quad (4)$$

we have (2). Therefore, by Lemma 1, the function Q is an even entire function of exponential type $\sigma\leqslant 1$. Further (see [16]),

$$\left(\frac{1}{z}\frac{d}{dz}\right)^{l} Q(z) = (-1)^l\int_0^1 t^{2l+m}\frac{J_{-m+l-1/2}(tz)}{(tz)^{-m+l-1/2}}h(t)\,dt, \quad l\in\{1;2;\ldots;m\}, \quad (5)$$

$$\left(\frac{1}{z}\frac{d}{dz}\right)^{m} Q(z) = (-1)^m\int_0^1 t^{3m}\sqrt{tz}J_{-1/2}(tz)h(t)\,dt = \sqrt{\frac{2}{\pi}}(-1)^m\int_0^1 t^{2m}\cos(tz)q(t)\,dt,$$

where $q(t):=t^m h(t)\in L^2(0;1)$. Furthermore, using (4) and (5), we get (3). In addition, according to the Paley-Wiener theorem, the function $\left(z^{-1}d/dz\right)^m Q(z)$ belongs to the space $PW^2_{1,+}$ (see also [16]). Therefore, the necessity has been proved.

Sufficiency. If all the conditions of the theorem hold, then from the formula for the inverse Fourier cosine transformation it follows that the function

$$q(t) = \sqrt{\frac{2}{\pi}}(-1)^m\frac{1}{t^{2m}}\int_0^{+\infty}\cos(tz)\left(\frac{1}{z}\frac{d}{dz}\right)^m Q(z)\,dz, \quad q(t)=t^m h(t),$$

belongs to the space $L^2(0;1)$, and

$$\left(\frac{1}{z}\frac{d}{dz}\right)^{m} Q(z) = (-1)^m\int_0^1 t^{3m}\sqrt{tz}J_{-1/2}(tz)h(t)\,dt.$$

Using (3) and consecutively applying the Fubini theorem $m-1$ times, we obtain

$$\frac{Q'(z)}{z} - \frac{Q'(z)}{z}\bigg|_{z=0} = -\int_0^1 t^{2+m}\left(\frac{J_{-m+1/2}(tz)}{(tz)^{-m+1/2}} - \frac{2^{m-1/2}}{\Gamma(-m+3/2)}\right)h(t)\,dt.$$

Since

$$\frac{Q'(z)}{z}\bigg|_{z=0} = -\frac{2^{m-1/2}}{\Gamma(-m+3/2)}\int_0^1 t^{2+m}h(t)\,dt,$$

we have

$$Q'(z) = -\int_0^1 t^{2+m} z\frac{J_{-m+1/2}(tz)}{(tz)^{-m+1/2}}h(t)\,dt.$$

Further, using (see [3, p. 11], [25, p. 349], [35, p. 45]) $(z^\nu J_{-\nu}(z))' = -z^\nu J_{-\nu+1}(z)$, $\nu \in \mathbb{R}$, we get $((tw)^{m+1/2}J_{-m-1/2}(tw))'_w = -t(tw)^{m+1/2}J_{-m+1/2}(tw)$. Furthermore, applying Fubini's theorem, we obtain

$$\begin{aligned} Q(z)-Q(0) &= -\int_0^1 t^{2+m}h(t)\,dt\int_0^z w\frac{J_{-m+1/2}(tw)}{(tw)^{-m+1/2}}\,dw \\ &= \int_0^1 t^m h(t)\,dt\int_0^z d\big((tw)^{m+1/2}J_{-m-1/2}(tw)\big) \\ &= \int_0^1 t^m\left(\frac{J_{-m-1/2}(tz)}{(tz)^{-m-1/2}} - \frac{2^{m+1/2}}{\Gamma(-m+1/2)}\right)h(t)\,dt. \end{aligned}$$

Hence, taking into account (2), we have (1). Thus, the theorem is proved. □

Let $\widetilde{E}_{2m,+}$ be the class of entire functions Q that can be presented in the form (1) with some function $h \in L^2((0;1);x^{2m}dx)$, $m \in \mathbb{N}$, and let $E_{2m,+}$ be the class of even entire functions Q of exponential type $\sigma \leqslant 1$ such that conditions (2), (3) are fulfilled with $h \in L^2((0;1);x^{2m}dx)$ and the function $\left(z^{-1}d/dz\right)^m Q(z)$ belongs to the space $PW^2_{1,+}$. Evidently, $\widetilde{E}_{2m,+} = E_{2m,+}$.

THEOREM 2. *Let $m \in \mathbb{N}$ and $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of nonzero complex numbers such that $\rho_k^2 \neq \rho_n^2$ for $k \neq n$. For a system $\{\psi_{k,m} : k \in \mathbb{N}\}$ to be incomplete in the space $L^2((0;1);x^{2m}dx)$ it is necessary and sufficient that a sequence $(\rho_k)_{k\in\mathbb{Z}\setminus\{0\}}$, where $\rho_{-k} := -\rho_k$, $k \in \mathbb{N}$, is a subsequence of zeros of some nonzero entire function $Q \in E_{2m,+}$.*

Proof. According to Hahn-Banach theorem (see, e.g., [17, p. 131], [18, p. 4258]), the system $\{\psi_{k,m} : k \in \mathbb{N}\}$ is incomplete in $L^2((0;1);x^{2m}dx)$ if and only if there exists a nonzero function $h \in L^2((0;1);x^{2m}dx)$ such that

$$\int_0^1 \rho_k^m\sqrt{x\rho_k}J_{-m-1/2}(x\rho_k)x^{2m}h(x)\,dx = 0, \quad m \in \mathbb{N},$$

for all $k \in \mathbb{N}$. Hence, taking into account Theorem 1, we obtain the required proposition. Theorem 2 is proved. □

THEOREM 3. *Let $m \in \mathbb{N}$ and $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of distinct nonzero complex numbers such that $|\Im\rho_k| \geqslant \delta|\rho_k|$ for all $k \in \mathbb{N}$ and some $\delta > 0$. If a system $\{\psi_{k,m} : k \in \mathbb{N}\}$ is complete in $L^2((0;1);x^{2m}dx)$, then*

$$\sum_{k=1}^{\infty}\frac{1}{|\rho_k|} = +\infty. \tag{6}$$

Proof. Suppose, to the contrary, that the system $\{\psi_{k,m} : k \in \mathbb{N}\}$ is not complete in $L^2((0;1);x^{2m}dx)$. Then, by Theorem 2, there exists a nonzero entire function $Q \in E_{2m,+}$ for which the sequence $(\rho_k)_{k\in\mathbb{Z}\setminus\{0\}}$ is a subsequence of zeros. Due to Lemma 1, we have $|Q(x)| \leqslant c_1(1+|x|)^m$ for all $x \in \mathbb{R}$ and $m \in \mathbb{N}$. This implies

$$\int_{-\infty}^{+\infty} \frac{\log^+ |Q(x)|}{1+x^2}\, dx < +\infty.$$

Therefore, by Lemma 2, we get

$$\sum_{k\in\mathbb{N}} \left| \Im \frac{1}{\rho_k} \right| < +\infty.$$

Since $|\Im \rho_k| \geqslant \delta |\rho_k|$, $\delta > 0$, for all $k \in \mathbb{N}$, and

$$\left| \Im \frac{1}{\rho_k} \right| = \frac{|\Im \rho_k|}{|\rho_k|^2} \geqslant \frac{\delta}{|\rho_k|},$$

we have

$$\sum_{k=1}^{\infty} \frac{1}{|\rho_k|} < +\infty.$$

This contradicts condition (6). Thus, the theorem is proved. □

THEOREM 4. *Let $m \in \mathbb{N}$ and $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of distinct nonzero complex numbers such that $\rho_k^2 \neq \rho_n^2$ for $k \neq n$. Let a sequence $(\rho_k)_{k\in\mathbb{Z}\setminus\{0\}}$, where $\rho_{-k} := -\rho_k$, be a sequence of zeros of some even entire function G of exponential type $\sigma \leqslant 1$ for which on the rays $\{z : \arg z = \varphi_j\}$, $j \in \{1;2;3;4\}$, $\varphi_1 \in [0;\pi/2)$, $\varphi_2 \in [\pi/2;\pi)$, $\varphi_3 \in (\pi;3\pi/2]$, $\varphi_4 \in (3\pi/2;2\pi)$, we have*

$$|G(z)| \geqslant c_2(1+|z|)^m e^{|\Im z|}. \tag{7}$$

Then the system $\{\psi_{k,m} : k \in \mathbb{N}\}$ is complete in $L^2((0;1);x^{2m}dx)$.

Proof. Assume the converse. Then, according to Theorem 2, there exists a nonzero even entire function $Q \in E_{2m,+}$ for which the sequence $(\rho_k)_{k\in\mathbb{Z}\setminus\{0\}}$ is a subsequence of zeros. Let $V(z) = Q(z)/G(z)$. Then V is an even entire function of order $\tau \leqslant 1$, for which by Lemma 1, we obtain

$$|V(z)| \leqslant c_3 \frac{1}{\sqrt{1+|\Im z|}}, \quad \arg z = \varphi_j, \quad j \in \{1;2;3;4\}.$$

Therefore, according to the Phragmén-Lindelöf theorem (see [17, p. 38], [18, p. 4263]), we get $V(z) \equiv 0$. Hence $Q(z) \equiv 0$. This contradiction proves the theorem. □

COROLLARY 1. ([20]) *Let $m \in \mathbb{N}$ and $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of zeros of the function $J_{-m-1/2}$. Then the system $\{\psi_{k,m} : k \in \mathbb{N}\}$ is complete in $L^2((0;1);x^{2m}dx)$.*

Proof. Indeed, a sequence $(\rho_k)_{k\in\mathbb{Z}\setminus\{0\}}$, where $\rho_{-k} := -\rho_k$, is a sequence of zeros of an even entire function $G(z) := z^{m+1/2} J_{-m-1/2}(z)$, $m \in \mathbb{N}$, of exponential type $\sigma \leqslant 1$ (see [1], [18, p. 4252]). Since ([25, p. 352], [35, p. 199])

$$J_{-\nu}(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z + \frac{\pi}{2}\nu - \frac{\pi}{4}\right) + O\left(|z|^{-3/2} e^{|\Im z|}\right), \quad \nu \in \mathbb{R},$$

as $z \to \infty$ and $|\arg z| < \pi$, we have (7) (see also [20]). Therefore, by Theorem 4, the system $\left\{\psi_{k,m} : k \in \mathbb{N}\right\}$ is complete in $L^2((0;1); x^{2m}dx)$. Corollary 1 is proved. □

THEOREM 5. *Let $m \in \mathbb{N}$ and $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of distinct nonzero complex numbers such that $\rho_k^2 \neq \rho_n^2$ for $k \neq n$. Let a sequence $(\rho_k)_{k\in\mathbb{Z}\setminus\{0\}}$, where $\rho_{-k} := -\rho_k$, be a sequence of zeros of some even entire function $G \notin E_{2m,+}$ of exponential type $\sigma \leqslant 1$ for which on the rays $\{z : \arg z = \varphi_j\}$, $j \in \{1;2;3;4\}$, $\varphi_1 \in [0;\pi/2)$, $\varphi_2 \in [\pi/2;\pi)$, $\varphi_3 \in (\pi;3\pi/2]$, $\varphi_4 \in (3\pi/2;2\pi)$, the inequality*

$$|G(z)| \geqslant c_4 (1+|z|)^{-\alpha} e^{|\Im z|}$$

holds with $\alpha < 2-m$. Then the system $\left\{\psi_{k,m} : k \in \mathbb{N}\right\}$ is complete in $L^2((0;1); x^{2m}dx)$.

Proof. Assume the converse. Then, according to Theorem 2, there exists a nonzero even entire function $Q \in E_{2m,+}$ for which the sequence $(\rho_k)_{k\in\mathbb{Z}\setminus\{0\}}$ is a subsequence of zeros. Let $V(z) = Q(z)/G(z)$. Then V is an even entire function of order $\tau \leqslant 1$, for which by Lemma 1, we get

$$|V(z)| \leqslant c_5 \frac{(1+|z|)^{\alpha+m}}{\sqrt{1+|\Im z|}}, \quad \arg z = \varphi_j, \quad j \in \{1;2;3;4\}.$$

Since $\alpha + m < 2$, according to the Phragmén-Lindelöf theorem, the function V is a polynomial of degree $\zeta < 2$. However, V is an even entire function, and therefore the function V is a constant. Hence, $Q(z) = c_6 G(z)$ and $Q \notin E_{2m,+}$. Thus, we have a contradiction and the proof of the theorem is completed. □

THEOREM 6. *Let $m \in \mathbb{N}$ and $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of distinct nonzero complex numbers such that $\rho_k^2 \neq \rho_n^2$ for $k \neq n$. Let a sequence $(\rho_k)_{k\in\mathbb{Z}\setminus\{0\}}$, where $\rho_{-k} := -\rho_k$, be a sequence of zeros of some even entire function $F \notin E_{2m,+}$ of exponential type $\sigma \leqslant 1$ such that for some $\alpha < 2-m$ and $\eta \in \mathbb{R}$*

$$|F(x+i\eta)| \geqslant \delta |x|^{-\alpha}, \quad \delta > 0, \quad |x| > 1. \tag{8}$$

Then the system $\left\{\psi_{k,m} : k \in \mathbb{N}\right\}$ is complete in $L^2((0;1); x^{2m}dx)$.

Proof. Let $m \in \mathbb{N}$, $F \notin E_{2m,+}$ and the inequality (8) is true. Suppose, to the contrary, that the system $\left\{\psi_{k,m} : k \in \mathbb{N}\right\}$ is not complete in $L^2((0;1); x^{2m}dx)$. Then, by Theorem 2, there exists a nonzero even entire function $Q \in E_{2m,+}$ which vanishes at the points ρ_k. However, the sequence $(\rho_k)_{k\in\mathbb{Z}\setminus\{0\}}$, where $\rho_{-k} := -\rho_k$, is a sequence of zeros of an even entire function $F \notin E_{2m,+}$ of exponential type $\sigma \leqslant 1$. Therefore, $T(z) = Q(z)/F(z)$ is an even entire function of order $\tau \leqslant 1$. Since $Q \in E_{2m,+}$, taking into account Lemma 1, we obtain

$$|Q(x+i\eta)| \leqslant c_7 \frac{e^{|\eta|}}{\sqrt{1+|\eta|}} \left(1+\sqrt{x^2+\eta^2}\right)^m, \quad x \in \mathbb{R}.$$

Using (8), we get

$$|T(x+i\eta)| \leqslant c_8(1+|x|)^{m+\alpha}, \quad x \in \mathbb{R}.$$

In view of this, we have that $T(z)$ is a polynomial of degree $\zeta < 2$. Further, since T is an even entire function, we have $T(z) = c_9$. Furthermore, $F(z) = c_{10}Q(z)$ and $F \in E_{2m,+}$. This contradiction concludes the proof of the theorem. □

Let $n(t)$ be the number of points of the sequence $(\rho_k)_{k\in\mathbb{N}} \subset \mathbb{C}$ satisfying the inequality $|\rho_k| \leqslant t$, i.e., $n(t) := \sum_{|\rho_k|\leqslant t} 1$, and let

$$N(r) := \int_0^r \frac{n(t)}{t}\, dt, \quad r > 0.$$

THEOREM 7. *Let $m \in \mathbb{N}$ and $(\rho_k)_{k\in\mathbb{N}}$ be an arbitrary sequence of distinct nonzero complex numbers. If*

$$\limsup_{r\to+\infty} \left(N(r) - \frac{2r}{\pi} + \frac{1}{2}\log r - m\log(1+r) \right) = +\infty,$$

then the system $\left\{\psi_{k,m} : k \in \mathbb{N}\right\}$ is complete in $L^2((0;1);x^{2m}dx)$.

Proof. It suffices to assume the incompleteness of the system $\left\{\psi_{k,m} : k \in \mathbb{N}\right\}$ and prove that

$$\limsup_{r\to+\infty} \left(N(r) - \frac{2r}{\pi} + \frac{1}{2}\log r - m\log(1+r) \right) < +\infty. \tag{9}$$

By virtue of Theorem 2, there exists a nonzero even entire function $Q \in E_{2m,+}$ of exponential type $\sigma \leqslant 1$ for which the sequence $(\rho_k)_{k\in\mathbb{N}}$ is a subsequence of zeros. We may consider that $Q(0) = 1$. Then, consecutively applying the Jensen formula (see [17, p. 10], [18, p. 4316]) and Lemma 1, we obtain

$$\begin{aligned}
N(r) &\leqslant \frac{1}{2\pi}\int_0^{2\pi} \log|Q(re^{i\varphi})|\, d\varphi \\
&\leqslant c_{11} + \frac{1}{2\pi}\int_0^{2\pi} \left(r|\sin\varphi| - \frac{1}{2}\log(1+r|\sin\varphi|) + m\log(1+r) \right) d\varphi \\
&\leqslant c_{11} + \frac{1}{2\pi}\int_0^{2\pi} \left(r|\sin\varphi| - \frac{1}{2}\log r - \frac{1}{2}\log|\sin\varphi| + m\log(1+r) \right) d\varphi \\
&= \frac{2r}{\pi} - \frac{1}{2}\log r + m\log(1+r) + c_{12}, \quad r > 0,
\end{aligned}$$

whence it follows (9). The theorem is proved. □

THEOREM 8. *Let $m \in \mathbb{N}$ and $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of distinct nonzero complex numbers. Assume that $|\rho_k| \leqslant \Delta k + \beta + \alpha_k$ for $0 < \Delta < \frac{\pi}{2+\pi(m-1)}$, $-\Delta < \beta < 1 - \Delta\left(\frac{2}{\pi} + m\right)$, and a sequence $(\alpha_k)_{k\in\mathbb{N}}$ such that $\alpha_k \geqslant 0$, $\alpha_k = O(1)$ as $k \to +\infty$ and*

$$\sum_{k=1}^{\infty} |\alpha_{k+1} - \alpha_k| < +\infty, \qquad \sum_{k=1}^{\infty} \frac{\alpha_k}{k} < +\infty.$$

Then the system $\left\{\psi_{k,m} : k \in \mathbb{N}\right\}$ is complete in $L^2((0;1);x^{2m}dx)$.

Proof. Let $\mu_k = \Delta k + \beta + \alpha_k$, $k \in \mathbb{N}$, and

$$n_1(t) = \sum_{\mu_k \leqslant t} 1, \quad N_1(r) = \int_0^r \frac{n_1(t)}{t}\,dt, \quad r > 0.$$

Then $n(t) \geqslant n_1(t)$, $N(r) \geqslant N_1(r)$ and $n_1(t) = n$ for $\Delta n + \beta + \alpha_n \leqslant t < \Delta(n+1) + \beta + \alpha_{n+1}$ ($n_1(t) = 0$ on $(0;\mu_1)$). Let $r \in [\mu_s;\mu_{s+1})$. Then $s = \frac{r}{\Delta} + O(1)$ as $r \to +\infty$. Therefore, under the assumptions of the theorem, by analogy with [8, 9] (see also [13, 15]), we obtain

$$\begin{aligned}
N_1(r) &\geqslant \sum_{k=1}^{s-1} k \log \frac{\Delta(k+1)+\beta}{\Delta k + \beta} + O(1) \\
&\quad - \left| \sum_{k=1}^{s-1} k \left(\log \frac{\Delta(k+1)+\beta+\alpha_{k+1}}{\Delta k + \beta + \alpha_k} - \log \frac{\Delta(k+1)+\beta}{\Delta k + \beta} \right) \right| \\
&\geqslant \frac{r}{\Delta} - \left(\frac{1}{2} + \frac{\beta}{\Delta}\right) \log r - c_{13} \sum_{k=1}^{\infty} \left(|\alpha_{k+1} - \alpha_k| + \frac{\alpha_k}{k} \right) + O(1) \\
&\geqslant \frac{r}{\Delta} - \left(\frac{1}{2} + \frac{\beta}{\Delta}\right) \log r + O(1), \quad r \to +\infty.
\end{aligned}$$

In view of this, for $m \in \mathbb{N}$, $0 < \Delta < \frac{\pi}{2+\pi(m-1)}$ and $-\Delta < \beta < 1 - \Delta\left(\frac{2}{\pi} + m\right)$, we get

$$\begin{aligned}
&\limsup_{r\to+\infty} \left(N(r) - \frac{2r}{\pi} + \frac{1}{2}\log r - m\log(1+r) \right) \\
&\quad\geqslant \limsup_{r\to+\infty} \left(N_1(r) - \frac{2r}{\pi} + \frac{1}{2}\log r - m\log(1+r) \right) \\
&\quad\geqslant \limsup_{r\to+\infty} \left(\frac{r}{\Delta} - \left(\frac{1}{2} + \frac{\beta}{\Delta}\right)\log r - \frac{2r}{\pi} + \frac{1}{2}\log r - m\log(1+r) + O(1) \right) \\
&\quad\geqslant \limsup_{r\to+\infty} \left(r\left(\frac{1}{\Delta} - \frac{2}{\pi}\right) - \left(\frac{\beta}{\Delta} + m\right)\log(1+r) + O(1) \right) \\
&\quad\geqslant \limsup_{r\to+\infty} \left(r\left(\frac{1}{\Delta} - \frac{2}{\pi} - \frac{\beta}{\Delta} - m\right) + O(1) \right) = +\infty.
\end{aligned}$$

Finally, according to Theorem 7, we obtain the required proposition. The proof of theorem is completed. $\square$

REMARK 1. In the case $m = 1$, Theorems 1–8 has been proved in [13], and for $m = 2$ in [15].

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