AN ANALOGUE OF THE PALEY-WIENER THEOREM FOR THE HANKEL TYPE TRANSFORM IN WEIGHTED L^2-SPACES

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ABSTRACT. In this paper, we establish some analogues of the Paley-Wiener theorem for the Hankel type transform of half-integer order less than -1 in weighted L^2-spaces. These Paley-Wiener-type theorems give a description of the class of even entire functions of exponential type $\sigma \leq 1$ under this transformation in terms of solutions from the corresponding spaces of some differential equations.

1. INTRODUCTION

Let $L^2(X)$ be the space of all measurable functions $f : X \to \mathbb{C}$ on a measurable set $X \subseteq \mathbb{R}$ with the norm

$$\|f\|_{L^2(X)}^2 := \int_X |f(x)|^2 \, dx,$$

let $\gamma \in \mathbb{R}$ and let $L^2((0;1); t^\gamma dt)$ be the weighted Lebesgue space of all measurable functions $f : (0;1) \to \mathbb{C}$, satisfying

$$\int_0^1 t^\gamma |f(t)|^2 \, dt < +\infty.$$

The function h belongs to the space $L^2((0;1); t^\gamma dt)$ if and only if the function $g(t) = t^{\gamma/2} h(t)$ belongs to $L^2(0;1)$. Let (see, for example, [4, p. 4], [33, p. 40], [1], [16])

$$J_\nu(z) = \sum_{k=0}^\infty \frac{(-1)^k (z/2)^{\nu+2k}}{k! \Gamma(\nu + k + 1)}, \quad z \in \mathbb{C},$$

be the Bessel function of the first kind of index $\nu \in \mathbb{R}$, where Γ is the classical Gamma function. Since (see [33, p. 55])

$$J_{-m-1/2}(z) = \sqrt{\frac{2}{\pi}} z^{m+1/2} \left(\frac{d}{zdz}\right)^m \left(\frac{\cos z}{z}\right), \quad m \in \mathbb{N},$$

the function $f(t) = z^m \sqrt{tz} J_{-m-1/2}(tz)$ belongs to the space $L^2((0;1); t^{2m}dt)$ for every $m \in \mathbb{N}$ and $z \in \mathbb{C}$.

An entire function G is said to have an exponential type $\sigma \in [0; +\infty)$ (see [15, p. 4], [17, p. 4262]) if for any $\varepsilon > 0$ there exists a constant $c(\varepsilon)$ such that $|G(z)| \le c(\varepsilon) \exp((\sigma + \varepsilon)|z|)$ for all $z \in \mathbb{C}$. Let us denote by PW_σ^2 the set of all entire functions of exponential type $\sigma \in (0; +\infty)$ whose narrowing on $\mathbb{R}$ belongs to $L^2(\mathbb{R})$, and by $PW_{\sigma,+}^2$ we denote the class of even entire functions from PW_σ^2. By the Paley-Wiener theorem (see [15, p. 69], [17, p. 4263], [35, p. 12]), the class PW_σ^2 coincides with the class of functions G representable as

$$G(z) = \int_{-\sigma}^{\sigma} e^{itz} g(t)\, dt, \quad g \in L^2(-\sigma; \sigma),$$

and the class $PW_{\sigma,+}^2$ consists of the functions G of the following kind:

$$G(z) = \int_0^{\sigma} \cos(tz) g(t)\, dt, \quad g \in L^2(0; \sigma).$$

Moreover, $\|g\|_{L^2(0;\sigma)} = \sqrt{2/\pi} \|G\|_{L^2(0;+\infty)}$ and

$$g(t) = \frac{2}{\pi} \int_0^{+\infty} G(z) \cos(tz)\, dz.$$

The Hankel transform of order $\nu \ge -1/2$ of a function $f \in L^2(0; +\infty)$ is defined by the formula (see [20, 21, 23, 33])

$$f(z) = \int_0^{+\infty} \sqrt{tz} J_\nu(tz) h(t)\, dt, \quad h \in L^2(0; +\infty),$$

where the integral is taken in the L^2-sense or in the mean, that is,

$$\lim_{u \to +\infty} \int_0^{+\infty} \left| f(z) - \int_0^u \sqrt{tz} J_\nu(tz) h(t)\, dt \right|^2 dz = 0.$$

By the Plancherel theorem for the Hankel transform (see [20, p. 212], [21]), we have $\|f\|_{L^2(0;+\infty)} = \|h\|_{L^2(0;+\infty)}$ and

$$h(t) = \int_0^{+\infty} \sqrt{tz} J_\nu(tz) f(z)\, dz, \quad f \in L^2(0; +\infty), \quad \nu > -1.$$

As is known, the Hankel transforms are generalizations of the Fourier transforms and it is natural to ask whether such a representation for entire functions is possible in this case also. The L^2-analogue of the Paley-Wiener theorem for Hankel transforms of order $\nu \ge -1/2$ was established in the papers [2, 6, 21, 22, 23, 34, 36]. In particular, Griffith [6] (see also [2, 23, 36]) has been prove the following statement.

Theorem 1.1 ([6]). *Let $\nu \ge -1/2$. A function G has the representation*

$$G(z) = z^{-\nu} \int_0^1 \sqrt{t} J_\nu(zt) g(t)\, dt$$

with $g \in L^2(0; 1)$ if and only if it is an even entire function of exponential type $\sigma \le 1$ such that $z^{\nu+1/2} G(z) \in L^2(0; +\infty)$.

Griffith's result has been extended by Andersen and de Jeu [3], Betancor, Linares and Méndez [5], Tuan and Zayed [21, 22], Unni [23], Weiss [34] and Zemanian [36]. In addition, in the article [27] (see also [7, 8, 26, 28]), Vynnyts'kyi and the author generalized Theorem 1.1 to Hankel transforms of order $\nu \geq 1/2$ in weighted spaces $L^2((0;1); t^{2p}dt)$, $p \in \mathbb{R}$.

Theorem 1.2 ([27]). *Let $\nu \geq 1/2$ and $p \in \mathbb{R}$. An entire function Ω has the representation*

$$\Omega(z) = z^{-\nu} \int_0^1 \sqrt{t} J_\nu(tz) t^{p-1} h(t) \, dt$$

with some function $h \in L^2((0;1); x^{2p}dx)$ if and only if it is an even entire function of exponential type $\sigma \leq 1$ such that $z^{-\nu+1/2}(z^{2\nu}\Omega(z))' \in L^2(0;+\infty)$. In this case,

$$h(t) = t^{-p} \int_0^{+\infty} \sqrt{tz} J_{\nu-1}(tz) z^{-\nu+1/2}(z^{2\nu}\Omega(z))' \, dz.$$

Besides, using Theorem 1.1, in the papers [26, 28, 29, 30] were found a criteria of the completeness, minimality and basicity of the system $\left\{\sqrt{x\rho_k} J_\nu(x\rho_k) : k \in \mathbb{N}\right\}$ in $L^2(0;1)$ if $\nu \geq -1/2$ and $(\rho_k)_{k \in \mathbb{N}}$ is an arbitrary sequence of distinct nonzero complex numbers. Moreover, taking into account Theorem 1.2, in [7, 8, 27] were established a necessary and sufficient conditions for the completeness and minimality of the system $\left\{x^{-p-1}\sqrt{x\rho_k} J_\nu(x\rho_k) : k \in \mathbb{N}\right\}$ in $L^2((0;1); x^{2p}dx)$, where $\nu \geq 1/2$, $p \in \mathbb{R}$ and $(\rho_k)_{k \in \mathbb{N}}$ is an arbitrary sequence of nonzero complex numbers. Those results are formulated in terms of sequences of zeros of functions from certain classes of entire functions.

Approximation properties of the systems of Bessel functions of index $\nu < -1$, $\nu \notin \mathbb{Z}$, in weighted L^2-spaces were investigated in the papers [10, 11, 12, 18, 19, 24, 25, 31, 32]. To study the completeness of a system $\left\{\rho_k^m \sqrt{x\rho_k} J_{-m-1/2}(x\rho_k) : k \in \mathbb{N}\right\}$ in the space $L^2((0;1); x^{2m}dx)$ where $m \in \mathbb{N}$ and $(\rho_k)_{k \in \mathbb{N}}$ is a sequence of distinct nonzero complex numbers, we need to obtain an analogue of the Paley-Wiener theorem for the Hankel type transform of half-integer order less than -1 in this weighted space. Finding for analogue of Theorem 1.2 for Hankel type transforms of arbitrary order $\nu < -1$, $\nu \in \mathbb{R}$, often faces certain difficulties and this problem remains open.

In the present paper, we study the class of the Hankel type transforms of half-integer order less than -1 of even entire functions Q of exponential type $\sigma \leq 1$ of the kind

$$Q(z) = z^m \int_0^1 \sqrt{tz} J_{-m-1/2}(tz) t^{2m} h(t) \, dt, \quad m \in \mathbb{N}, \qquad (1.1)$$

with some function $h \in L^2((0;1); x^{2m}dx)$. This class of integral transforms arises in the investigation of some non-classical boundary value problems for Bessel equation (see [9, 10, 11, 12, 13, 18, 19, 24, 25, 31, 32]) for which the system of their canonical eigenfunctions is over-complete. In the case where $m \in \{1; 2; 3\}$, an integral representation of a class of entire functions of this type were investigated in [9, 12, 13, 24, 25]. The conditions for the existence of this integral transforms

are formulated in terms of certain solutions from the corresponding spaces of some differential equations.

The aim of this paper is to obtain some analogues of the Paley-Wiener theorem for the Hankel type transform (1.1) in the weighted space $L^2((0;1); x^{2m}dx)$ (see Theorems 2.1 and 2.4). Close assertions can be found in [21, 22].

2. Main results

Our main results are the following statements.

Theorem 2.1. *Let $m \in \mathbb{N}$. An entire function Q has the representation (1.1) with some function $h \in L^2((0;1); x^{2m}dx)$ if and only if the differential equation*

$$(-2m + 1)f(z) + zf'(z) = Q(z) \tag{2.1}$$

has a solution $f = G$ which can be presented in the form

$$G(z) = z^{m-1} \int_0^1 \sqrt{tz} J_{-m+1/2}(tz) t^{2m-1} h(t)\, dt. \tag{2.2}$$

Proof. Necessity. Let $m \in \mathbb{N}$ and the function Q admits representation (1.1) with some function $h \in L^2((0;1); x^{2m}dx)$. Since (see [4, p. 12], [33, p. 45])

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z), \quad \nu \in \mathbb{R}, \tag{2.3}$$

we have

$$\begin{aligned}
Q(z) =& z^m \int_0^1 \sqrt{tz} J_{-m-1/2}(tz) t^{2m} h(t)\, dt \\
=& z^m \int_0^1 \sqrt{tz} \left(\frac{2(-m+1/2)J_{-m+1/2}(tz)}{tz} - J_{-m+3/2}(tz) \right) t^{2m} h(t)\, dt \\
=& (-2m+1) \int_0^1 \frac{J_{-m+1/2}(tz)}{(tz)^{-m+1/2}} t^m h(t)\, dt \\
& - z^{m+1/2} \int_0^1 J_{-m+3/2}(tz) t^{2m+1/2} h(t)\, dt.
\end{aligned}$$

Let

$$G(z) := \int_0^1 \frac{J_{-m+1/2}(tz)}{(tz)^{-m+1/2}} t^m h(t)\, dt.$$

Then

$$G(z) = z^{m-1} \int_0^1 \sqrt{tz} J_{-m+1/2}(tz) t^{2m-1} h(t)\, dt.$$

Since (see [4, p. 11], [33, p. 45])

$$\left(\frac{J_\nu(z)}{z^\nu} \right)' = -\frac{J_{\nu+1}(z)}{z^\nu}, \quad \nu \in \mathbb{R}, \tag{2.4}$$

we get

$$G'(z) = -\int_0^1 \frac{J_{-m+3/2}(tz)}{z^{-m+1/2}} t^{2m+1/2} h(t)\, dt.$$

Therefore, we obtain $(-2m+1)G(z) + zG'(z) = Q(z)$, that is, the function (2.2) is a solution to the equation (2.1). The necessity has been proved.

Sufficiency. Let $m \in \mathbb{N}$ and the equation (2.1) has a solution $f = G$ representing in the form (2.2) with some function $h \in L^2((0;1); x^{2m}dx)$. We have

$$f(z) = z^{m-1} \int_0^1 \sqrt{tz} J_{-m+1/2}(tz) t^{2m-1} h(t)\, dt = \int_0^1 \frac{J_{-m+1/2}(tz)}{(tz)^{-m+1/2}} t^m h(t)\, dt.$$

Using (2.3) and (2.4), we obtain

$$\begin{aligned}
Q(z) =& (-2m+1)f(z) + zf'(z) \\
=& (-2m+1)z^{m-1/2} \int_0^1 J_{-m+1/2}(tz) t^{2m-1/2} h(t)\, dt \\
& - z^{m+1/2} \int_0^1 J_{-m+3/2}(tz) t^{2m+1/2} h(t)\, dt \\
=& z^m \int_0^1 \sqrt{tz} \left((-2m+1)\frac{J_{-m+1/2}(tz)}{tz} - J_{-m+3/2}(tz) \right) t^{2m} h(t)\, dt \\
=& z^m \int_0^1 \sqrt{tz} J_{-m-1/2}(tz) t^{2m} h(t)\, dt.
\end{aligned}$$

Hence, the function Q has the form (1.1) with $h \in L^2((0;1); x^{2m}dx)$. Theorem 2.1 is proved. $\qquad\square$

Remark 2.2. In the case $m = 1$, Theorem 2.1 has been proved in [24, p. 11], in the case $m = 2$ in [9, p. 3], and for $m = 3$ in [13, p. 75]. Theorem 2.1 indicate the method of finding a description of the corresponding class of functions Q of the kind (1.1) for an arbitrary $m \in \mathbb{N}$ if the solution to the equation (2.1) is known for some m.

Example 2.3. Let $\alpha := 4(-\pi + 2)/\pi^3$. The function

$$\begin{aligned}
Q(z) =& -4\sqrt{\frac{2}{\pi}} \frac{z^2}{(z^2 - \pi^2/4)^3} \left(2z^2 \cos z + (z^2 - \pi^2/4)(z \sin z + 3 \cos z) \right) \\
& + \sqrt{\frac{2}{\pi}} \frac{z^2 \cos z - 7z \sin z - 15 \cos z}{z^2 - \pi^2/4} (1 - \alpha(z^2 - \pi^2/4))
\end{aligned}$$

can be represented in the form (1.1) with $m = 3$ and

$$h(t) = \frac{4}{t^5 \pi^3} \left(2 - 2\cos\left(\frac{\pi}{2}t\right) - \pi t \sin\left(\frac{\pi}{2}t\right) \right).$$

Indeed, the function

$$\begin{aligned}
G(z) =& \sqrt{\frac{2}{\pi}} z \frac{(1 - \alpha(z^2 - \pi^2/4))(z^2 - \pi^2/4)\sin z + 2z \cos z}{(z^2 - \pi^2/4)^2} \\
& + 3\sqrt{\frac{2}{\pi}} \frac{\cos z}{z^2 - \pi^2/4} (1 - \alpha(z^2 - \pi^2/4))
\end{aligned}$$

is represented in the form (2.2) with $m = 3$ (see [9, Example 1, p. 4]) and is a solution to the equation (2.1) for $m = 3$, because

$$G'(z) = -2\sqrt{\frac{2}{\pi}}\frac{z}{(z^2 - \pi^2/4)^3}\left(4z^2\cos z + (z^2 - \pi^2/4)(\cos z + 2z\sin z)\right)$$

$$+ \sqrt{\frac{2}{\pi}}\frac{-2\sin z + z\cos z}{z^2 - \pi^2/4}(1 - \alpha(z^2 - \pi^2/4)).$$

Therefore, according to Theorem 2.1, the function Q admits representation (1.1).

Let $[x]$ be the integer part of a real number x, and let $(z^{-1}d/dz)^m$ be the m-th power of the differential operator $z^{-1}d/dz$.

Theorem 2.4. *Let $m \in \mathbb{N}$. An entire function Q has the representation (1.1) with some function $h \in L^2((0;1); x^{2m}dx)$ if and only if the differential equation*

$$\sqrt{\frac{2}{\pi}}\sum_{s=0}^{[m/2]}\frac{(m+2s)!}{(2s)!(m-2s)!2^{2s}}z^{m-2s}f^{(m-2s)}(z)$$

$$-\sqrt{\frac{2}{\pi}}\sum_{s=0}^{[(m-1)/2]}\frac{(m+2s+1)!}{(2s+1)!(m-2s-1)!2^{2s+1}}z^{m-2s-1}f^{(m-2s-1)}(z) = Q(z) \tag{2.5}$$

has a solution $f = F$ belonging to the space $PW_{1,+}^2$. Moreover, the function $(z^{-1}d/dz)^m Q(z)$ also belongs to $PW_{1,+}^2$ and h can be found by one of the following formulas

$$h(t) = \frac{2}{\pi}\frac{1}{t^m}\int_0^{+\infty}F(z)\cos(tz)\,dz, \tag{2.6}$$

$$h(t) = \sqrt{\frac{2}{\pi}}(-1)^m\frac{1}{t^{3m}}\int_0^{+\infty}\cos(tz)\left(\frac{1}{z}\frac{d}{dz}\right)^m Q(z)\,dz. \tag{2.7}$$

Proof. Necessity. Let the function Q admits the representation (1.1) with some function $h \in L^2((0;1); x^{2m}dx)$, $m \in \mathbb{N}$, and

$$F(z) = \int_0^1\cos(tz)g(t)\,dt, \quad g \in L^2(0;1). \tag{2.8}$$

Since (see [19, 32], [33, pp. 46, 55])

$$\left(\frac{1}{z}\frac{\partial}{\partial z}\right)^s\frac{J_{-\nu}(tz)}{(tz)^{-\nu}} = (-1)^s t^{2s}\frac{J_{-\nu+s}(tz)}{(tz)^{-\nu+s}}, \quad t, \nu \in \mathbb{R}, \quad s \in \mathbb{N},$$

$$\sqrt{z}J_{-1/2}(z) = \sqrt{\frac{2}{\pi}}\cos z,$$

we get

$$\left(\frac{1}{z}\frac{d}{dz}\right)^s Q(z) = (-1)^s\int_0^1 t^{2s+m}\frac{J_{-m-1/2+s}(tz)}{(tz)^{-m-1/2+s}}h(t)\,dt,$$

for $s \in \{1; 2; \ldots; m\}$, and

$$
\left(\frac{1}{z}\frac{d}{dz}\right)^m Q(z)
$$

$$
= (-1)^m \int_0^1 t^{3m}\sqrt{tz}J_{-1/2}(tz)h(t)\,dt \tag{2.9}
$$

$$
= \sqrt{\frac{2}{\pi}}(-1)^m \int_0^1 t^{2m}\cos(tz)g(t)\,dt,
$$

where $g(t) := t^m h(t) \in L^2(0; 1)$. According to the Paley-Wiener theorem, the functions $F(z)$ and $(z^{-1}d/dz)^m Q(z)$ belong to $PW^2_{1,+}$. Further, since

$$
F^{(m-2s)}(z) = (-1)^s \int_0^1 t^{m-2s}\cos\left(tz + \frac{m\pi}{2}\right)g(t)\,dt, \quad s \in \{0; 1; \ldots; [m/2]\},
$$

$$
F^{(m-2s-1)}(z) = (-1)^s \int_0^1 t^{m-2s-1}\sin\left(tz + \frac{m\pi}{2}\right)g(t)\,dt,
$$

where $s \in \{0; 1; \ldots; [(m-1)/2]\}$, and for $m \in \mathbb{N}$ (see [33, p. 55])

$$
\sqrt{\frac{\pi}{2}}(tz)^{m+1/2}J_{-m-1/2}(tz)
$$

$$
= \cos\left(tz + \frac{m\pi}{2}\right)\sum_{s=0}^{[m/2]}\frac{(-1)^s(m+2s)!}{(2s)!(m-2s)!2^{2s}}(zt)^{m-2s} \tag{2.10}
$$

$$
- \sin\left(tz + \frac{m\pi}{2}\right)\sum_{s=0}^{[(m-1)/2]}\frac{(-1)^s(m+2s+1)!}{(2s+1)!(m-2s-1)!2^{2s+1}}(zt)^{m-2s-1},
$$

we obtain

$$
\sqrt{\frac{2}{\pi}}\sum_{s=0}^{[m/2]}\frac{(m+2s)!}{(2s)!(m-2s)!2^{2s}}z^{m-2s}F^{(m-2s)}(z)
$$

$$
-\sqrt{\frac{2}{\pi}}\sum_{s=0}^{[(m-1)/2]}\frac{(m+2s+1)!}{(2s+1)!(m-2s-1)!2^{2s+1}}z^{m-2s-1}F^{(m-2s-1)}(z)
$$

$$
=\sqrt{\frac{2}{\pi}}\int_0^1 \cos\left(tz + \frac{m\pi}{2}\right)\sum_{s=0}^{[m/2]}\frac{(-1)^s(m+2s)!}{(2s)!(m-2s)!2^{2s}}(zt)^{m-2s}g(t)\,dt
$$

$$
-\sqrt{\frac{2}{\pi}}\int_0^1 \sin\left(tz + \frac{m\pi}{2}\right)\sum_{s=0}^{[(m-1)/2]}\frac{(-1)^s(m+2s+1)!}{(2s+1)!(m-2s-1)!2^{2s+1}}(zt)^{m-2s-1}g(t)\,dt
$$

$$
=\int_0^1 (tz)^{m+1/2}J_{-m-1/2}(tz)t^m h(t)\,dt
$$

$$
=z^m \int_0^1 \sqrt{tz}J_{-m-1/2}(tz)t^{2m}h(t)\,dt = Q(z).
$$

Therefore, the necessity has been proved.

Sufficiency. Let the equation (2.5) has a solution $f = F$ belonging to $PW^2_{1,+}$.
Due to the Paley-Wiener theorem, we have

$$f(z) = \int_0^1 \cos(tz) g(t)\, dt, \quad g \in L^2(0; 1).$$

Hence, using (2.10), we get

$$Q(z) = \sqrt{\frac{2}{\pi}} \sum_{s=0}^{[m/2]} \frac{(m+2s)!}{(2s)!(m-2s)!2^{2s}} z^{m-2s} f^{(m-2s)}(z)$$

$$- \sqrt{\frac{2}{\pi}} \sum_{s=0}^{[(m-1)/2]} \frac{(m+2s+1)!}{(2s+1)!(m-2s-1)!2^{2s+1}} z^{m-2s-1} f^{(m-2s-1)}(z)$$

$$= z^m \int_0^1 \sqrt{tz} J_{-m-1/2}(tz) t^{2m} h(t)\, dt,$$

where $h(t) := t^{-m} g(t) \in L^2((0;1); x^{2m} dx)$. Thus, we obtain representation (1.1).
Formulas (2.6) and (2.7) follow from the equalities (2.8) and (2.9) and the formula
for the inverse Fourier cosine-transformation. Theorem 2.4 is proved.   □

Corollary 2.5. *Let an entire function Q be defined by the formula (1.1). Then Q is an even entire function of exponential type $\sigma \leq 1$ such that for all $z \in \mathbb{C}$ and $m \in \mathbb{N}$, we have*

$$|Q(z)| \leq c_1 \frac{e^{|\operatorname{Im} z|}}{\sqrt{1 + |\operatorname{Im} z|}} (1 + |z|)^m, \quad c_1 > 0.$$

Proof. Indeed, using (2.10) and Schwartz's inequality, similarly to [19] (see also
[9, 12, 15, 18, 24, 25]), we obtain the required statement.   □

Example 2.6. The function $Q(z) = \sqrt{2/\pi} z^8 \cos z$ cannot be represented in the
form (1.1) with $m = 4$. In fact, for this function Q and $m = 4$ the differential
equation (2.5) has the form

$$z^4 f^{(4)}(z) - 10 z^3 f^{(3)}(z) + 45 z^2 f''(z) - 105 z f'(z) + 105 f(z) = z^8 \cos z$$

and its solution is the function $F(z) = -6z^3 \sin z + 15 z \sin z + z^4 \cos z - 15 z^2 \cos z + C_1 z^7 + C_2 z^5 + C_3 z^3 + C_4 z$. But there are no constants C_1, C_2, C_3 and C_4 for
which the function F belongs to $PW^2_{1,+}$. Indeed, F is an even entire function only
if $C_1 = C_2 = C_3 = C_4 = 0$, and in this case, the function $G(z) = -6z^3 \sin z + 15 z \sin z + z^4 \cos z - 15 z^2 \cos z$ does not belong to $W^2_{1,+}$ since $G \notin L^2(\mathbb{R})$. Therefore,
the equation (2.5) with $Q(z) = \sqrt{2/\pi} z^8 \cos z$ and $m = 4$ has no solution belonging
to $PW^2_{1,+}$. Hence, by Theorem 2.4, the function Q cannot be represented in the
form (1.1) with $m = 4$.

Remark 2.7. In the case $m = 1$, Theorem 2.4 has been proved in [24, p. 11], in
the case $m = 2$ in [9, p. 4], and for $m = 3$ in [13, p. 76]. In the case $m = 1$, an
analog of Theorem 2.4 by some other method has been proved in [12, p. 6338].

Remark 2.8. Theorems 2.1 and 2.4 give a characterization of a class of even entire
functions Q of exponential type $\sigma \leq 1$, for which differential equations (2.1) and

(2.5) have certain solutions in the corresponding spaces. Similar problems are studied in many papers (see, for instance, [14]).

3. Conclusions

In this paper, we construct certain analogues of the Paley-Wiener theorem for the Hankel type transform of half-integer order less than -1 in weighted L^2-spaces. These Paley-Wiener-type theorems give a description of the class of even entire functions of exponential type $\sigma \leq 1$ under this transformation in terms of the existence of certain solutions of some differential equations (see Theorems 2.1 and 2.4). The corresponding examples are given (see Examples 2.3 and 2.6). Those results can be used for the investigation of completeness of the system $\left\{\rho_k^m \sqrt{x\rho_k} J_{-m-1/2}(x\rho_k) : k \in \mathbb{N}\right\}$ in $L^2((0;1); x^{2m}dx)$, where $m \in \mathbb{N}$ and $(\rho_k)_{k\in\mathbb{N}}$ is a sequence of distinct nonzero complex numbers.

4. Acknowledgements

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