Drohobych Ivan Franko State Pedagogical University

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SELECTED QUESTIONS OF MATHEMATICAL ANALYSIS (THE POSITIVE FUNCTIONS, DIRICHLET SERIES)

EDUCATIONAL AND METHODICAL MANUAL

Drohobych
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Дрогобицький державний педагогічний університет імені Івані Франка

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ВИБРАНІ ПИТАННЯ МАТЕМАТИЧНОГО АНАЛІЗУ (ДОДАТНІ ФУНКЦІЇ, РЯДИ ДІРІХЛЕ)

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Selected Questions of Mathematical Analysis (The Positive Functions, Dirichlet Series) / Educational and methodical manual [for students and lecturers of mathematical specialties]. Yu. Gal, O. Kutnyak. Drohobych: Drohobych Ivan Franko State Pedagogical University, 2024. 142 p.

The educational and methodological manual is written in accordance with the programme of the course "Selected Questions of Mathematical Analysis (The Positive Functions, Dirichlet Series)" for the field of study *01 Education/ Pedagogy* of the programme subject area *014 Secondary Education (Mathematics)*, approved by the Academic Council of Drohobych Ivan Franko State Pedagogical University (protocol No. 7 of August 30, 2022). It covers some classes of positive functions, the classical scale of their growth; the concept of the maximum modulus, maximum term, order and type of the entire function; representation of entire and analytical functions in the half plane by the Dirichlet series.

The manual is recommended for the students and the lecturers of the mathematical specialties of the University.

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Ю. Галь, О. Кутняк.
Вибрані питання математичного аналізу (Додатні функції, ряди Діріхле) / Навчально-методичний посібник [для студентів та викладачів математичних спеціальностей]. Ю. Галь, О. Кутняк. Дрогобич: Дрогобицький державний педагогічний університет імені Івані Франка, 2024. 142 с.

Навчально-методичний посібник написано відповідно до програми курсу «Вибрані питання математичного аналізу (Додатні функції, ряди Діріхле)» для галузі знань *01 Освіта/ Педагогіка* спеціальності *014 Середня освіта (Математика)*, затвердженої вченою радою Дрогобицького державного педагогічного університету імені Івані Франка (протокол № 7 від 30 серпня 2022 р.). У ньому висвітлено деякі класи додатних функцій, класична шкала їхнього зростання; поняття максимуму модуля, максимального члена, порядку і типу цілої функції; зображення цілих та аналітичних функцій у півлощині рядами Діріхле.

Рекомендований студентам та викладачам математичних спеціальностей університету.

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Зміст

Preface

The basis of the educational and methodological manual is based on the lectures that the authors for many years had read for the students of the Drohobych Ivan Franko State Pedagogical University.

Despite the completeness of the Theory of Analytical Functions, some of its subsections (theory of entire, meromorphic, subharmonious functions represented by different series) require significant researches, which need the study of the properties of positive functions.

To understand the text of the manual, you need to familiarize with complex numbers, algebraic operations on them and to know the basics of Mathematical Analysis and the Theory of Analytical Functions.

In the chapter "Positive functions" students have the opportunity to recall the concept of the limit of the function, to get acquainted with some classes of positive functions, as well as with the classical scale of their growth.

In the chapter "Entire functions" such concepts as maximum modulus, maximum term, order and type of the entire function given by the power series, zeros and the expanding of the function into the infinite product are characterized.

In the chapter "Dirichlet Series" the representation by the Dirichlet series the entire and analytical in a half-plane functions as generalizations of power series are considered.

Theoretical material is widely illustrated by the solvable examples, tasks for independent work, control questions.

The manual will be interested for those students who plan to study in the magistracy and postgraduate studies, as well as for teachers for the group work.

Передмова

В основу навчально-методичного посібника покладено лекції, які автори протягом багатьох років читали для студентів Дрогобицького державного педагогічного університету імені Івана.

Незважаючи на завершеність теорії аналітичних функцій, деякі її підрозділи (теорія цілих, мероморфних, субгармонійних функцій, зображених тими чи іншими рядами) вимагають істотних досліджень, що часто зводяться до вивчення властивостей додаткових функцій.

Щоб зрозуміти текст посібника, необхідно ознайомитися з комплексними числами, алгебраїчними діями над ними, а також знати основи математичного аналізу та теорії аналітичних функцій.

У розділі «Додатні функції» студенти мають можливість пригадати поняття границі функції, ознайомитися з деякими класами додатних функцій, а також з класичною шкалою їх зростання.

У розділі «Цілі функції» охарактеризовані такі поняття, як максимум модуля, максимальний член, порядок і тип цілої функції, заданої степеневим рядом, нулі і розвинення функції у нескінченний добуток.

У розділі «Ряди Діріхле» розглядається зображення цілих та аналітичних у півплощині функцій рядами Діріхле як узагальненнями степеневих рядів.

Теоретичний матеріал широко ілюструється розв’язними прикладами, завданнями для самостійної роботи, контрольними питаннями.

Посібник буде цікавий студентам, які планують навчатися в магістратурі та аспірантурі, а також учителям для гурткової роботи.

Section I. Positive functions

The upper and lower limits of the function

Let $\varphi : R \to R$ be an arbitrary function. If there is the sequence (x_n) approaching to $+\infty$ such that $\varphi(x_n) \to p$, when $n \to \infty$, then the number p is called the partial limit of the function φ when $x \to +\infty$.

Suppose that the set $\{p\}$ of partial limits of the function φ is bounded. Then there are the finite numbers $M = \sup\{p\}$ and $m = \inf\{p\}$. Let's prove that $M \in \{p\}$ and $m \in \{p\}$. To do this, suppose the opposite, that $M \notin \{p\}$, i. e. M is not a partial limit of the function φ. Then there doesn't exist the sequence $(x_n) \to +\infty$ such that $\varphi(x_n) \to p$ when $n \to \infty$. It means that there exists a number ε that for all $x \geq x_0(\varepsilon)$ takes place $\varphi(x) \notin (M - \varepsilon, M + \varepsilon)$. It follows that the interval $(M - \varepsilon, M + \varepsilon)$ does not contain the partial limits of the function φ. But it is impossible according to the definition of the number M.

So, if the set $\{p\}$ of partial limits of a function φ is bounded, then it has the largest and smallest elements. The largest of the partial limits is called the upper limit and is denoted by one of the symbols

$$\overline{\lim_{x \to +\infty}} \varphi(x), \quad \limsup_{x \to +\infty} \varphi(x),$$

and the smallest – the lower limit and is denoted by one of the symbols

$$\underline{\lim_{x \to +\infty}} \varphi(x), \quad \liminf_{x \to +\infty} \varphi(x).$$

If the set of partial limits is unbounded above or below, then we accept accordingly

$$\overline{\lim_{x \to +\infty}} \varphi(x) = +\infty, \qquad \underline{\lim_{x \to +\infty}} \varphi(x) = -\infty.$$

It is clear that the definition of the upper limit can be represented as follows:

$$\overline{\lim_{x\to+\infty}}\varphi(x)=A\neq\infty\equiv\begin{cases}1)\ (\exists(x_n)\to+\infty)\{\varphi(x_n)\to A\},\\ 2)\ (\forall\varepsilon>0)(\forall x\geq x_0(\varepsilon))\{\varphi(x)<A+\varepsilon\},\end{cases}$$

property 1) can be written as

$$(\forall\varepsilon>0)(\forall x)(\exists x>x_0)\{\varphi(x)>A-\varepsilon\}.$$

Similarly

$$\underline{\lim_{x\to+\infty}}\varphi(x)=a\neq\infty\equiv\begin{cases}1)\ (\exists(x_n)\to+\infty)\{\varphi(x_n)\to a\},\\ 2)\ (\forall\varepsilon>0)(\forall x\geq x_0(\varepsilon))\{\varphi(x)>a-\varepsilon\},\end{cases}$$

and property 1) can be written in the form of

$$(\forall\varepsilon>0)(\forall x_0)(\exists x>x_0)\{\varphi(x)<a+\varepsilon\}.$$

If $A=-\infty$ then $\overline{\lim_{x\to+\infty}}\varphi(x)=\underline{\lim_{x\to+\infty}}\varphi(x)=-\infty$ and if $A=+\infty$ then $\overline{\lim_{x\to+\infty}}\varphi(x)=+\infty$. It means that $(\exists(x_n)\to+\infty)\{\varphi(x_n)\to+\infty\}$. Similarly, if $a=+\infty$ then $\underline{\lim_{x\to+\infty}}\varphi(x)=\overline{\lim_{x\to+\infty}}\varphi(x)=+\infty$ and if $a=-\infty$ then $\underline{\lim_{x\to+\infty}}\varphi(x)=-\infty$. In this case $(\exists(x_n)\to+\infty)\{\varphi(x_n)\to-\infty\}$.

It is not difficult to see how these definitions will change, when $x\to b\neq+\infty$.

Note some properties of the upper and lower limits.

Lemma 1.1. *Inequalities*

$$\underline{\lim_{x\to+\infty}}\varphi_1(x)+\overline{\lim_{x\to+\infty}}\varphi_2(x)\leq\overline{\lim_{x\to+\infty}}(\varphi_1(x)+\varphi_2(x))\leq\overline{\lim_{x\to+\infty}}\varphi_1(x)+\overline{\lim_{x\to+\infty}}\varphi_2(x)\tag{1.1}$$

and

$$\underline{\lim_{x\to+\infty}}\varphi_1(x)+\underline{\lim_{x\to+\infty}}\varphi_2(x)\leq\underline{\lim_{x\to+\infty}}(\varphi_1(x)+\varphi_2(x))\leq\underline{\lim_{x\to+\infty}}\varphi_1(x)+\overline{\lim_{x\to+\infty}}\varphi_2(x)\tag{1.2}$$

are always hold, except the cases, when in the right or left parts (1.1) and (1.2), there is the uncertainty $+\infty+(-\infty)$.

Let's prove, for example, the inequalities (1.1). Let

$$\underline{\lim_{x\to+\infty}}\varphi_j(x)=a_j\,,\ \overline{\lim_{x\to+\infty}}\varphi_j(x)\leq A_j\quad j=1,2$$

and for simplicity we will assume that A_j and a_j are the finite numbers.

Then

$$(\forall\varepsilon>0)(\exists x_0)(\forall x\geq x_0)\{\varphi_j(x)<A_j+\varepsilon\},$$

i.e.

$$(\forall\varepsilon>0)(\exists x_0)(\forall x\geq x_0)\{\varphi_1(x)+\varphi_2(x)<A_1+A_2+2\varepsilon\}$$

and hence,

$$\overline{\lim_{x\to+\infty}}(\varphi_1(x)+\varphi_2(x))\leq A_1+A_2+2\varepsilon,$$

where, due to the arbitrariness of ε, we get the first inequality (1.1).

From the definition A_2 follows the existence of the sequence (x_n), which tends to the $+\infty$, that $\varphi_2(x_n)>A_2-\varepsilon$. Inequality also takes place for this sequence

$$\underline{\lim_{n\to+\infty}}\varphi_1(x_n)\geq\underline{\lim_{x\to+\infty}}\varphi_1(x)=a_1,$$

that is, $\varphi_1(x_n)>a_1-\varepsilon$ for enough large n. Thus, there is a sequence (x_n^*) that tends to $+\infty$, that $\varphi_1(x_n^*)+\varphi_2(x_n^*)>A_2+a_1-2\varepsilon$. From this, we receive the inequality

$$\overline{\lim_{x\to+\infty}}(\varphi_1(x)+\varphi_2(x))\geq\overline{\lim_{n\to+\infty}}(\varphi_1(x_n^*)+\varphi_2(x_n^*))\geq A_2+a_1-2\varepsilon,$$

and due the arbitrariness of ε, we have the second inequality (1.1).

Note that in the cases, where uncertainty $+\infty+(-\infty)$ is in the right or left parts (1.1) and (1.2), these inequalities may not hold. To prove that it is enough to take $\varphi_1(x)=\varphi(x)$ and $\varphi_2(x)=b-\varphi(x)$, where $\varphi(x)\to+\infty$, when $x\to+\infty$. Then $\lim\limits_{x\to+\infty}\varphi_1(x)=+\infty$, $\lim\limits_{x\to+\infty}\varphi_2(x)=-\infty$ and $\lim\limits_{x\to+\infty}(\varphi_1(x)+\varphi_2(x))=b$, where b is the arbitrary number.

From Lemma 1.1 it follows: if there exists $\lim\limits_{x\to+\infty}\varphi_1(x)$, then

$$\overline{\lim_{x\to+\infty}}(\varphi_1(x)+\varphi_2(x))=\lim_{x\to+\infty}\varphi_1(x)+\overline{\lim_{x\to+\infty}}\varphi_2(x)$$

and

$$\underline{\lim_{x\to+\infty}}(\varphi_1(x)+\varphi_2(x))=\lim_{x\to+\infty}\varphi_1(x)+,\underline{\lim_{x\to+\infty}}\varphi_2(x)$$

except the cases of uncertainty $+\infty+(-\infty)$.

The correctness of the next four lemmes follows either from the definition of the upper and lower limits or from the considerations similar to those used in the proof of Lemma 1.1.

Lemma 1.2. *The next equalities are true*

$$\overline{\lim_{x\to+\infty}}(-\varphi(x)) = -\underline{\lim_{x\to+\infty}}\varphi(x),$$

$$\underline{\lim_{x\to+\infty}}(-\varphi(x)) = -\overline{\lim_{x\to+\infty}}\varphi(x).$$

Lema 1.3. *Let* $\varphi_j(x) \geq 0 \ (x > x_0)$, $j = 1, 2$. *The inequalities*

$$\underline{\lim_{x\to+\infty}}\varphi_1(x)\,\overline{\lim_{x\to+\infty}}\varphi_2(x) \leq \overline{\lim_{x\to+\infty}}(\varphi_1(x)\varphi_2(x)) \leq \overline{\lim_{x\to+\infty}}\varphi_1(x)\overline{\lim_{x\to+\infty}}\varphi_2(x)$$

and

$$\underline{\lim_{x\to+\infty}}\varphi_1(x)\underline{\lim_{x\to+\infty}}\varphi_2(x) \leq \underline{\lim_{x\to+\infty}}(\varphi_1(x)\varphi_2(x)) \leq \underline{\lim_{x\to+\infty}}\varphi_1(x)\,\overline{\lim_{x\to+\infty}}\varphi_2(x)$$

always hold except the cases, when the uncertainty $0\times\infty$ is in their right or left parts.

In the case of existence of one of the limits in the Lemma 1.3 the inequality is converted into equality.

Lemma 1.4. *If* f *is the continuous non-decreasing function, then*

$$\overline{\lim_{x\to+\infty}} f(\varphi(x)) = f\left(\overline{\lim_{x\to+\infty}}\varphi(x)\right),$$

and if f *is continuous non-increasing function, then*

$$\underline{\lim_{x\to+\infty}} f(\varphi(x)) = f\left(\underline{\lim_{x\to+\infty}}\varphi(x)\right).$$

Let $\varphi: R \to R$ be an arbitrary function. Let's define

$$\Phi_*(x) = \inf\{\varphi(t): t \geq x\}, \quad \Phi^*(x) = \sup\{\varphi(t): t \geq x\}.$$

It is easy to see that $\Phi_*(x)$ is non-decreasing function and $\Phi^*(x)$ is non-increasing function. Therefore, there exists the limits $\lim_{x\to\infty}\Phi_*(x) = b$ and $\lim_{x\to+\infty}\Phi^*(x) = B$, finite or infinite. Let's define $\underline{\lim_{x\to+\infty}}\varphi(x) = a$ and $\overline{\lim_{x\to+\infty}}\varphi(x) = A$.

Lemma 1.5. *The next equalities are true* $a = b$ *and* $A = B$.

Let's prove, for example, that $A = B$. It follows from the definition B that $\varphi(x) \leq \Phi^*(x)$, i. e. $A \leq B$. Let's assume the opposite, that $A < B$. Then, if $A = -\infty$, then $\varphi(x) \to -\infty$, when $x \to +\infty$. Therefore, $\Phi^*(x) \to -\infty$, when $x \to +\infty$, i.e, $B = -\infty$. But it is impossible. If $-\infty < A < +\infty$, then for chosen C, $A < C < B$ there is x_0, that is

$\varphi(x) \le C$ for all $x \ge x_0$. Hence, $\Phi^*(x_0) \le C$ and Φ^* is the non-increasing function. Then $B \le C$, which is impossible.

Convex functions

Let the function f be defined on the interval (a,b) and $a < x_1 < x_2 < b$. We connect the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ of the plane by a segment. If for any points x_1, x_2, such the segment is located above the curve $y = f(x)$, then the function f is called the convex one on the interval (a,b), and if it located under this curve, it is called the concave function on (a,b). We will study the convex functions, because a concave functions have the similar properties.

The equation of the segment connecting the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$, has the form

$$\frac{y - f(x_1)}{f(x_2) - f(x_1)} = \frac{x - x_1}{x_2 - x_1},$$

i.e.

$$y = \frac{x_2 - x}{x_2 - x_1} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2).$$

Therefore the convexity of the function f on the interval (a,b) means that the next inequality takes place

$$f(x) \le \frac{x_2 - x}{x_2 - x_1} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2) \tag{1.3}$$

for each three numbers $a < x_1 < x < x_2 < b$.

Finally, let in (1.3) $x = (1-t)x_1 + tx_2,\ 0 < t < 1$. Then $x \in (x_1, x_2)$ and the convexity of the function f on (a,b) means that the next inequality is hold

$$f((1-t)x_1 + tx_2) \le (1-t)f(x_1) + tf(x_2)$$

for all and $a < x_1 < x_2 < b$ and $0 < t < 1$.

Theorem 1.1. *The convex on (a,b) function f is continuous on (a,b).*

Proof. If in (1.3) we'll tend $x \downarrow x_1$, then we get the inequality $\varlimsup_{x \downarrow x_1} f(x) \le f(x_1)$, i. e. for each $x_0 \in (a,b)$ is hold

$$\varlimsup_{x \downarrow x_0} f(x) \le f(x_0). \tag{1.4}$$

If in (1.3) we'll tend $x_2 \downarrow x$, we get the inequality $\varliminf_{x_2 \downarrow x} f(x) \ge f(x)$, i. e. for each $x_0 \in (a,b)$ is hold

$$\varliminf_{x \downarrow x_0} f(x) \le f(x_0). \tag{1.5}$$

From inequalities (1.4) and (1.5) follow that $\lim_{x \downarrow x_0} f(x) = f(x_0)$ for all $x_0 \in (a,b)$, therefore f is continuous on the right at each point from (a,b). The proof of continuity on the left is similar. You only need to approach $x \uparrow x_2$ in (1.3) and then $x_1 \uparrow x$.

The theorem is proved.

The convex functions may be undifferentiated, such as the function $y = |x|$ at the point $x = 0$. However, such a theorem is correct.

Theorem 1.2. *The convex on the interval (a,b) function f has a non-decreasing continuous derivative on (a,b), except the countable set of points, where one-sided derivatives exist. At each such a point the left-sided derivative does not exceed the right-sided derivative.*

Proof. Let's first show that there exists a right-sided derivative $f'_+(x_0) = \lim_{x \downarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$ at every point $x_0 \in (a,b)$. Let's prove that $\frac{f(x) - f(x_0)}{x - x_0}$ is the non-decreasing function on (x_0, b). It means that for all points $x_0 < x < x_2 < b$ the next inequality is hold

$$\frac{f(x) - f(x_0)}{x - x_0} \le \frac{f(x_2) - f(x_0)}{x_2 - x_0}.$$

It is easy to verify that inequality (1.6) is equivalent to inequality (1.3) with $x_1 = x_0$ and therefore, the existence of a right-sided derivative at every point $x_0 \in (a,b)$ is proved.

The existence of $f'_-(x_0)$ at every point $x_0 \in (a,b)$ also follows from the inequality

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} \leq \frac{f(x) - f(x_0)}{x - x_0}, \quad a < x_1 < x < x_0, \tag{1.7}$$

which is also equivalent to inequality (1.3).

The inequality (1.3) is also equivalent to the inequality

$$\frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x_2) - f(x)}{x_2 - x}, \quad x_1 < x < x_2. \tag{1.8}$$

By tending $x_1 \uparrow x$ and $x_2 \downarrow x$, we get the inequality $f'_-(x) \leq f'_+(x)$ at every point $x \in (a,b)$.

Let's take four points $a < x_1 < x < x_0 < x_2 < b$. Then from (1.8) we have

$$\frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x_0) - f(x)}{x_0 - x} \leq \frac{f(x_2) - f(x_0)}{x_2 - x_0}. \tag{1.9}$$

If we tend in (1.9) $x \downarrow x_1$ and $x_2 \downarrow x_0$, then we'll get the inequality $f'_+(x_1) \leq f'_+(x_0)$, i. e. the right-sided derivative is a non-decreasing function and therefore it is continuous function on (a,b) except the countable set of points. If we tend in (1.9) $x_1 \uparrow x$ and $x_0 \uparrow x_2$, then similarly we'll get that the left-sided derivative is continuous on (a,b), except a countable set of points. Finally, if we tend in (1.9) $x \downarrow x_1$ and $x_0 \uparrow x_2$, we will have the inequality $f'_+(x_1) \leq f'_-(x_2)$, $x_1 < x_2$.

Let $(\alpha,\beta) \subset (a,b)$ be any interval, where f'_+ and f'_- are continuous. Then for arbitrary points $x_1 < x_0$ from (α,β), the inequality $f'_+(x_1) \leq f'_-(x_0) \leq f'_+(x_0)$ takes place, i.e.

$$f'_+(x_0) = \lim_{x \to x_0} f'_+(x_1) \leq f'_-(x_0) \leq f'_+(x_0),$$

it means that the function f has a derivative at each point from (α,β).

The theorem is proved.

Theorem 1.3. *If the function* f *is convex on the* (a,b) *and there exists the derivative* $f'(x_0)$ *at some point* $x_0 \in (a,b)$, *then for all* $x \in (a,b)$

$$f(x) - f(x_0) \geq f'(x_0)(x - x_0). \tag{1.10}$$

Proof. If in (1.6) we tend $x \downarrow x_0$, we'll get the inequality $f(x_2) - f(x_0) \geq f'(x_0)(x_2 - x_0)$, $x_2 > x_0$, i. e. the inequality (1.10) for $x > x_0$. For $x < x_0$ the inequality (1.10) similarly follows from (1.7), when $x \uparrow x_0$.

Theorem 1.4. *In order that the function* f *to be convex on* (a,b), *it is necessary and sufficient that*

$$f(x) = f(x_0) + \int_{x_0}^{x} \varphi(t)dt \ , \tag{1.11}$$

where x_0 *is an arbitrary fixed point from* (a,b) *and* φ *is the non-decreasing function on* (a,b).

Proof. If the function f is the convex on (a,b), then as has been shown when proving the theorem 1.2, it has a non-decreasing derivative except a countable set of points, at which this derivative has gaps of the first kind. Then there exists the integral

$$\int_{x_0}^{x} f'_+(t)dt = \int_{x_0}^{x} f'(t)dt = f(x) - f(x_0),$$

i. e. we have (1.11) from $\varphi(t) = f'_+(t)$.

On the contrary, since φ is the non-decreasing function, then from (1.11) for $x_1 < x < x_2$ we have

$$\frac{f(x) - f(x_1)}{x - x_1} = \frac{1}{x - x_1}\int_{x_1}^{x} \varphi(t)dt \leq \varphi(x) \leq \frac{1}{x_2 - x}\int_{x}^{x_2} \varphi(t)dt = \frac{f(x_2) - f(x)}{x_2 - x},$$

that is (1.8) is hold and therefore (1.3).

The theorem 1.4 is proved.

Corollary 1.1. *If the function* f *is convex on* $(a,+\infty)$, *then there exists*

$$\lim_{x\to+\infty}\frac{f(x)}{x}=k\in(-\infty,+\infty],$$

if $a<0$ *and* $f(0)=0$ *, then* $\frac{f(x)}{x}\uparrow k\ (x\uparrow+\infty)$.

Proof. Since from (1.11) the function φ is non-decreasing, therefore it has a limit $k\in(-\infty,+\infty]$, then according to the H'Lopital`s rule

$$\lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to+\infty}\varphi(x)=k\,.$$

If $a<0$ and $f(0)=0$, then from (1.6) when $x=0$ we have $f(x)/x\le f(x_2)/x_2$, $0<x<x_2<+\infty$.

The corollary is proved.

If the function $f(x)=F(e^x)$ is convex on $(\ln\alpha,\ln\beta)$, then the function F on (α,β), $\alpha\ge 0$ is called logarithmically convex on (α,β). Thus, a function F is logarithmically convex on (α,β) if and only if for any three numbers $x_1,x_2,x_3(\ln\alpha<x_1<x<x_2<\ln\beta)$ the inequality takes place

$$F(e^x)\le\frac{x_2-x}{x_2-x_1}F(e^{x_1})+\frac{x-x_1}{x_2-x_1}F(e^{x_2}).$$

If we make a replacement $x=\ln r$ it means that for any three numbers $r_1,r_2,r_3(\alpha<r_1<r<r_2<\beta)$ the inequality is hold

$$F(r)\le\frac{\ln r_2-\ln r}{\ln r_2-\ln r_1}F(r_1)+\frac{\ln r-\ln r_1}{\ln r_2-\ln r_1}F(r_2).$$

From the definition of a logarithmically convex function, it follows that the function is continuous, has a continuous derivative, except a countable number of points in which there are one-sided derivatives, and at each point the left-sided derivative does not exceed the right-sided derivative.

By the Theorem 1.4, the function F is logarithmically convex on (α,β) if and only if

$$F\left(e^{x}\right)=F\left(e^{x_0}\right)+\int_{x_0}^{x}\varphi(t)dt,$$

where x_0 is an arbitrary fixed point from $(\ln\alpha,\ln\beta)$, ad φ is non-decreasing function on $(\ln\alpha,\ln\beta)$. If we make a replacement $x=\ln r$, w'll get

$$F(r)=F(r_0)+\int_{\ln r_0}^{\ln r}\varphi(t)dt=F(r_0)+\int_{r_0}^{r}\frac{\varphi(\ln t)}{t}dt.$$

Hence, denoting $\psi(t)=\varphi(\ln t)$, we get that in order that the function F is logarithmically convex on (α,β), it is necessary and sufficient, that

$$F(r)=F(r_0)+\int_{r_0}^{r}\frac{\psi(t)}{t}dt,$$

where r_0 is an arbitrary fixed point from (α,β) and ψ is non-decreasing function on (α,β).

Slowly varying functions

The positive continuous function φ on $[a,+\infty)$ is called a slowly varying one, if

$$\frac{\varphi(cx)}{\varphi(x)}\to 1 \quad (x\to+\infty) \tag{1.12}$$

for all $c\in(0,+\infty)$. If the function is increasing, then it is called the slow-increasing.

Theorem 1.5 (Karamat`s). If the function φ is slowly varying, then the tending to the limit in (1.12) will be uniform for each c from the fixed interval $[c_1,c_2]$, $0<c_1<c_2<+\infty$.

Proof. Let`s $f(x)=\ln\varphi\left(e^{x}\right)$. Then (1.12) will write in the form

$$f(x+\lambda)-f(x)\to 0, \quad x\to+\infty \tag{1.13}$$

for all $\lambda\in R$ and, therefore, it is necessary to prove that the approaching to the limit in (1.13) is uniform for $\lambda\in[a,b]$, $-\infty<a<b<+\infty$. We can assume that $[a,b]=[0,1]$, otherwise, we can consider the function $\tilde{f}=f((b-a)x)$ and reduce $[a,b]$ to $[0,1]$. Then

$$f(x+\lambda)-f(x)=\tilde{f}(y+\mu)-\tilde{f}(y)+f(x+a)-f(x),$$

where $y=\dfrac{x+a}{b-a}$, $\mu=\dfrac{\lambda-a}{b-a}$, so $(x\to+\infty)\Leftrightarrow(y\to+\infty)$ and $(\lambda\in[a,b])\Leftrightarrow(\mu\in[0,1])$.

Suppose the opposite that the approaching to the limit in (1.13) is not uniform with $\lambda\in[a,b]$. Then

$$(\exists\varepsilon>0)(\exists(x_n)\uparrow+\infty)(\exists\{\mu_n\}\in[0,1])(\forall n\geq 1)\quad\{|f(x_n+\mu_n)-f(x_n)\geq\varepsilon|\}. \tag{1.14}$$

Let's denote

$$U_n=\{\mu\in[0,2]:(\forall m\geq n)\{|f(x_m+\mu)-f(x_m)<\varepsilon/2|\}\} \tag{1.15}$$

and

$$V_n=\{\lambda\in[0,2]:(\forall m\geq n)\{|f(x_m+\mu_n+\lambda)-f(x_m+\mu_n)<\varepsilon/2|\}\}. \tag{1.16}$$

The sets U_n and V_n, from (1.13) have the properties

$$U_{n-1}\subset U_n\uparrow[0,2],\quad V_{n-1}\subset V_n\uparrow[0,2]\quad(n\to\infty).$$

That's why $mesU_N>3/2$ and $mesV_N>3/2$ for enough large N. Let $V_N^*=V_N+\mu_N$ (to each element from V_N we add μ_n). Then $mesV_N^*>3/2$. Obviously that $U_n\subset[0,2]\subset[0,3]$ and $V_N^*\subset[0,3]$. Therefore, $U_n\cap V_N^*\neq\varnothing$, that is $(\exists\lambda_0\in U_N)$ and $\{\lambda_0\in V_N^*\}$ and, hence, $\lambda_0-\mu_N\in V_N$. Therefore, from (1.15) it follows that

$$|f(x_N+\lambda_0)-f(x_N)|<\varepsilon/2, \tag{1.17}$$

and from (1.16) we have

$$|f(x_N+\mu_N+\lambda_0-\mu_N)-f(x_N+\mu_N)|<\varepsilon/2,$$

i.e.

$$|f(x_N+\lambda_0)-f(x_N+\mu_N)|<\varepsilon/2. \tag{1.18}$$

From (1.17) and (1.18) we obtain

$$|f(x_N+\mu_N)-f(x_N)|=|f(x_N+\mu_N)-f(x_N+\lambda_0)+f(x_N+\lambda_0)-f(x_N)|<\varepsilon,$$

which is not possible because of (1.14).

Theorem 1.5 is proved.

Using theorem 1.5, we'll prove the theorem of the representation of a slowly varying function.

Theorem 1.6 (Karamat`s). *In order that the function* φ *defined on* $[a,+\infty)$ *to be slowly varying, it is necessary and sufficient that*

$$\varphi(x)=\varphi_0(x)\exp\left\{\int_a^x \frac{\delta(t)}{t}dt\right\}, \qquad (1.19)$$

where φ_0 *and* δ *are continuous functions* $\varphi_0(t)\to a_0>0$ *on* $[a,+\infty)$ *and* $\delta(t)\to 0$ *when* $t\to+\infty$.

Proof. Let the function φ be slowly varying one on $[1,+\infty)$. Let's denote $f(x)=\ln\varphi(e^x)$. By the Theorem 1.5 the approaching to the limit in (1.13) is uniform with $\lambda\in[0,1]$. Therefore

$$\int_0^1\{f(x+\lambda)-f(x)\}d\lambda\to 0 \quad (x\to+\infty),$$

i.e.

$$\varepsilon_1(x)=\int_x^{x+1} f(t)dt-f(x)\to 0 \quad (x\to+\infty),$$

$$f(x)=\int_x^{x+1} f(t)dt-\varepsilon_1(x)=\int_0^{x+1} f(t)dt-\int_0^x f(t)dt-\varepsilon_1(x)=$$

$$=\int_1^{x+1} f(t)dt-\int_0^x f(t)dt+\int_0^1 f(t)dt-\varepsilon_1(x)=\int_0^1 f(t)dt-\varepsilon_1(x)+\int_0^x\{f(t+1)-f(t)\}dt.$$

Let's denote

$$c(x)=\int_0^1 f(t)dt-\varepsilon_1(x), \quad \Delta(t)=f(t+1)-f(t),$$

so $c(x)\to c_0=\int_0^1 f(t)dt$, $\Delta(x)\to 0\ (x\to+\infty)$, and $\ln\varphi(e^x)=f(x)=c(x)+\int_0^x\Delta(t)dt$.

Therefore,

$$\varphi(x)=e^{c(\ln x)}\exp\left\{\int_0^{\ln x}\Delta(t)dt\right\}=e^{c(\ln x)}\exp\left\{\int_0^{\ln x}\frac{\Delta(\ln t)}{t}dt\right\}.$$

Denoting $\varphi_0(x)=e^{c(\ln x)}$ and $\delta(t)=\Delta(\ln t)$, we get (1.19).

On the contrary, for all $c\in(0,+\infty)$ from (1.19) we have

$$\frac{\varphi(cx)}{\varphi(x)}=\frac{\varphi_0(cx)}{\varphi_0(x)}\exp\left\{\int\limits_x^{cx}\delta(t)d\ln t\right\}=(1+o(1))\exp\{o(1)\ln c\}=(1+o(1))\quad(x\to+\infty),$$

i.e. the function φ is slowly variable.

Corollary 1.2. *If the function φ is a slow varying and $\eta>0$ is a fixed number, then $x^{-\eta}\varphi(x)\to 0$ and $x^{\eta}\varphi(x)\to+\infty$, when $x\to+\infty$.*

In fact, from (1.19) for each $\varepsilon\in(0,\eta)$, when $x\geq x_0(\varepsilon)$, we get

$$\varphi_0(x)\exp\{-\varepsilon\ln x\}\leq\varphi(x)\leq\varphi_0(x)\exp\{\varepsilon\ln x\},$$

from here, the necessary relations follow easily.

Let's give another important criterion for the slow varying.

Theorem 1.7. *In order that continuous function φ to be slowly varying, it is necessary and sufficient, that there exists the continuously differentiated function ψ, such that $\psi(x)\sim\varphi(x),\, x\to+\infty$ and*

$$\frac{x\psi'(x)}{\psi(x)}\to 0\quad x\to+\infty. \tag{1.20}$$

Proof. If the function φ is a slow varying, then it has the representation (1.19). Let's denote

$$\psi(x)=a_0\exp\left\{\int\limits_a^x\frac{\delta(t)}{t}dt\right\},$$

so,

$$\lim_{x\to+\infty}\frac{\varphi(x)}{\psi(x)}=\lim_{x\to+\infty}\frac{\varphi_0(x)}{a_0}=1$$

and

$$\frac{x\psi'(x)}{\psi(x)}=\frac{d\ln\psi'(x)}{\ln x}=\frac{d}{\ln x}\int\limits_a^x\delta(t)d\ln t=\delta(x)\to 0\quad(x\to+\infty).$$

On the contrary, if (1.2) takes place, then for all $c\in(0,+\infty)$

$$|\ln\psi(cx)-\ln\psi(x)|=\left|\int\limits_x cx\frac{\xi\psi'(x)}{\psi(x)}d\xi\right|=o(1)\ln c,\quad x\to+\infty,$$

i.e.

$$\lim_{x\to+\infty}\frac{\varphi(cx)}{\varphi(x)}=\lim_{x\to+\infty}\frac{\psi(cx)}{\psi(x)}=1.$$

The theorem is proved.

When proving Theorem 1.7, we have shown that for continuously differentiated functions from the condition (1.20) follows a slow change of function ψ. In the mathematical literature, condition (11) is often taken as the definition of a slow change. Using (1.20), it is easy to show that the function $\psi(x)=\ln_k x, k\ge 1$ is a slowly varying function (here $\ln_0 x = x, \ln_1 x = \ln x, \ln_k x = \ln(\ln_{k-1} x)$). Let's notice, that from the slow change of the function φ does not follow the condition (1.20) in general, as evidenced by the example of a continuously differentiable function $\varphi(x)=\ln x+\sin x$.

Function $\ln^+ x$

Let $D\subset R$ be the domain of definition of the function f. Let's denote $D^+=\{x\in D: f(x)>0\},\ D^-=\{x\in D: f(x)<0\}$ and

$$f^+(x)=\begin{cases} f(x), x\in D^+,\\ 0, x\in R\setminus D^+,\end{cases}\qquad f^-(x)=\begin{cases} -f(x), x\in D^-,\\ 0, x\in R\setminus D^-.\end{cases}$$

In particular, for $a\in R$

$$a^+=\frac{|a|+a}{2},\quad a^-=\frac{|a|-a}{2},\quad a=a^+-a^-,\quad |a|=a^++a^-,$$

and

$$\ln^+ x=\begin{cases}\ln x, x\ge 1,\\ 0, x<1.\end{cases}$$

Theorem 1.8. *If* $a_k\ge 0\ (k=1,2,\dots,n)$ *, then*

$$\ln^+\prod_{k=1}^n a_k\le\sum_{k=1}^n\ln^+ a_k\,, \tag{1.21}$$

and if $a_k>0\ (k=1,2,\dots,n)$*, then*

$$\ln^+ \sum_{k=1}^{n} a_k \le \sum_{k=1}^{n} \ln^+ a_k + \ln n$$

(1.22)

Proof. Let's start from (1.21). If $\prod_{k=1}^{n} a_k = 0$, then $\ln^+ \prod_{k=1}^{n} a_k = 0$ and the inequality (1.21) is obvious. If $\prod_{k=1}^{n} a_k \ne 0$, then $a_k > 0\ (k = 1,2,\ldots,n)$

$$\ln \prod_{k=1}^{n} a_k = \sum_{k=1}^{n} \ln a_k \le \sum_{k=1}^{n} \ln^+ a_k ,$$

that is, again leads to (1.21).

Using (1.21), we have

$$\ln^+ \sum_{k=1}^{n} a_k \le \ln^+ (n \max\{a_k : 1 \le k \le n\}) \le \ln^+ n + \ln^+ \max\{a_k : 1 \le k \le n\} \le \sum_{k=1}^{n} \ln^+ a_k + \ln n .$$

Semi-continuous functions

The function $f : E \to [-\infty;+\infty)$ defined on the set $E \subset R$ is called the upper semi-continuous at $x_0 \in E$, if for all $A > f(x_0)$ there exists the number $\delta = \delta(x_0, A) > 0$, such that the inequality $A > f(x)$ is hold for all $x \in (E \cap \{|x - x_0| < \delta\})$. The function f is called the upper semi-continuous one on the set E if f is the upper semi-continuous at each point of the set E. The function f is called the lower semi-continuous one (in x_0, on E) if the function $\varphi = -f$ is the lower semi-continuous (respectively in x_0, on E). We will study the the upper semi-continuous functions.

The simplest examples of the upper semi-continuous functions are: $f(x) = -\frac{1}{|x|}$ and $f(x) = \ln|x|,\ x \in R$. The properties of the upper semi-continuous function follows directly from its definition:

I. In order that the function $f: E \to [-\infty;+\infty)$ to be the upper semi-continuous on E, it is necessary and sufficient that the set $\{x \in E: f(x) < A\}$ is open on E for all $x \in R$.

II. The sum of two upper semi-continuous functions is the upper semi-continuous function.

III. The product of the upper semi-continuous function and the positive constant is the upper semi-continuous function.

IV. If the function f is the upper semi-continuous at the point x_0, then $\overline{\lim\limits_{x \to x_0}} f(x) \le f(x_0)$.

V. If the function f is the upper and the lower semi-continuous at the point x_0, then f is continuous at this point.

The semi-continuous functions have some properties close to the properties of the continuous functions. For example, the analogue of Weierstrass`s classical theorem is correct.

Theorem 1.9. *If the function f is the upper semi-continuous on the compact K, then it reaches its maximum on K, i. e. there exists $x_0 \in K$, such that $f(x) \le f(x_0)$ for all $x \in K$.*

Proof. Let`s denote $M = \sup\{f(x): x \in K\}$. If $f(x) \equiv -\infty$, then $M \equiv -\infty$ and we can take any point in the role x_0.

If $f(x) \ne -\infty$, then there exists a sequence (x_k) on K, such that $f(x) \to M$. Since K is the compact, we can assume that $x_k \to \xi \in K$ and we need to show that $f(\xi) = M$. To do this, suppose from the opposite, that $f(\xi) = \lambda < M$ and we'll choose $\lambda < \mu < M$. Then, by property I, the set $G(\mu) = \{x \in K: f(x) < \mu\}$ is open on K and contains the point ξ. It follows that in some interval $(\xi - \delta, \xi + \delta) \subset G(\mu)$ there are infinitely much members of the sequence (x_k), and we have $M > \mu > f(x_k) \to M, k \to \infty$.

Theorem 1.10. *If* (f_n) *is the non-increasing sequence of the upper semi-continuous functions on the set* E *, then the function* $f(x)=\lim_{n\to\infty} f_n(x)$ *is the upper semi-continuous function on* E *.*

Proof. It is clear that $f(x)\le f(x_1)<+\infty$ on E. Let`s denote $E(A)=\{x\in E: f(x)<A\}$. We need to prove that $E(A)$ is the open set.

Let's take an arbitrary point $\xi\in E(A)$. Since $\psi_n(\xi)\downarrow f(\xi)<A$, then $\psi_n(\xi)<A$ for all $n\ge n_0$. Therefore, the sets $E_n(A)=\{x\in E: f_n(x)<A\}$ are the neighbourhoods of the point ξ. But $f(x)\le f_n(x)$, that is $E_n(A)\subset E(A)$. Hence, $E(A)$ contains the neighbourhood of the point ξ and is the open set.

Theorem 1.11. *If* f *is the upper semi-continuous function on the set* E *, then there exists a decreasing sequence* (f_n) *of continuous functions on* E *, such that* $f(x)=\lim_{n\to\infty} f_n(x)$.

Proof. Let's consider only the case, when E is the compact.

We can assume that $f(x)>-\infty$ for all $x\in E$, because otherwise we can consider the function $\exp\{f(x)\}$, which is the semi-continuous on the set E and positive. If $\varphi_n(x)\downarrow \exp\{f(x)\}, n\to\infty$ and all φ_n are continuous functions, then all φ_n are the positive functions and functions $f_n(x)=\ln(1/n+\varphi_n(x))$ are continuous on E and $f_n(x)\downarrow f(x), n\to\infty$.

So, let it E be a compact and $f(x)>-\infty$ for all $x\in E$. Let's denote

$$f_n(x)=\max\{f(y)-n|x-y|: y\in E\},$$

and let $y_n=y_n(x)$ such that

$$f_n(x)=f(y_n(x))-n|x-y_n(x)|.$$

Let's show that each f_n is a continuous function on E. In fact, for everyone $x^*\in E$

$$f_n(x)\ge f(y_n(x^*))-n|x-y_n(x^*)|\ge$$

$$\geq f\left(y_n\left(x^*\right)\right)-n\left|x^*-y_n\left(x^*\right)\right|-n\left|x-x^*\right|=f_n\left(x^*\right)-n\left|x-x^*\right|$$

and similarly,

$$f_n\left(x^*\right)\geq f_n(x)-n\left|x-x^*\right|,$$

i.e.

$$\left|f_n\left(x^*\right)-f_n(x)\right|\geq n\left|x-x^*\right|,$$

from where the continuity of the function f_n easily follows.

It is clear that the sequence (f_n) is decreasing. Suppose that $f_n(x)\downarrow g(x), n\to\infty$. Since $f_n(x)\geq f(x)>-\infty$, then $g(x)\geq f(x)>-\infty$. If we take the limit in (1.23), we'll get $y_n(x)\to x, n\to\infty$ and therefore by to the property IV

$$\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty}\left(f\left(y_n(x)\right)-n\left|x-y_n(x)\right|\right)\leq\overline{\lim_{n\to\infty}}\, f\left(y_n(x)\right)\leq f(x),$$

i.e. $g(x)=f(x)$. Therefore, $g(x)=f(x)$, the Theorem 1.11 is proved.

Order, type and category of growth

Let the function $A=A(r)$ be defined on $[0,+\infty)$, positive and non-decreasing on $[r_0,+\infty), r_0\geq 0$.

The values

$$\rho=\overline{\lim_{r\to+\infty}}\frac{\ln^+ A(r)}{\ln r},\quad \lambda=\underline{\lim_{r\to+\infty}}\frac{\ln^+ A(r)}{\ln r}$$

are called the order $\rho=\rho[A]$ and lower order $\lambda=\lambda[A]$ of the function A correspondly.

In the case, when $0\leq\rho<+\infty$, to characterize the growth of the function A we use the values of the type $T=T[A]$ and the lower type $\tau=\tau[A]$ by the formulas

$$T=\overline{\lim_{r\to+\infty}}\frac{A(r)}{r^\rho},\quad \tau=\underline{\lim_{r\to+\infty}}\frac{\ln^+ A(r)}{\ln r^\rho}.$$

If $T=0$, then A has the minimum (zero) type, if $0<T<+\infty$, then A has the normal (average) type, if $T=+\infty$, then – the maximum (infinite) type.

Suppose that $r_0 = 1$, i. e. the function $A = A(r)$ is positive and non-decreasing on $[1,+\infty)$. Suppose also that $\rho = \rho[A] < +\infty$ and $\mu > \rho$. Then $A(r) \le r^{\gamma}$ for every $\gamma \in (\rho, \mu)$, when $r \ge r_0(\gamma)$, so

$$\int_1^{\infty} \frac{A(r)}{r^{\mu+1}} dr < +\infty . \tag{1.23}$$

On the other hand, if (1.23) takes place, then for each $\varepsilon > 0$ when $r \ge r_0(\varepsilon)$

$$\varepsilon > \int_r^{\infty} \frac{A(t)}{t^{\mu+1}} dt \ge A(r) \int_r^{\infty} \frac{1}{t^{\mu+1}} dt = \frac{1}{\mu r^{\mu}} ,$$

i.e.

$$A(r) < \varepsilon \mu r^{\mu} \; r \ge r_0(\varepsilon).$$

(1.24)

From (1.23) and (1.24) it follows that

$$\rho[A] = \inf \left\{ \mu > 0 : \int_1^{\infty} \frac{A(r)}{r^{\mu+1}} dr < +\infty \right\},$$

Moreover, if the integral in (1.23) is divergent for all $\mu > 0$, then we suppose that $\rho[A] = +\infty$.

From (1.24) it also follows that $\rho[A] \le \mu$ and if $\rho[A] = \mu$, then A has the minimum type, that is $T[A] = 0$.

Let`s suppose $\rho = \rho[A] < +\infty$ again. Let`s consider that the function A belongs to the class of convergence or divergence depending on whether the integral is convergent or divergent

$$\int_1^{\infty} \frac{A(r)}{r^{\rho+1}} dr < +\infty . \tag{1.25}$$

As mentioned, if the function A belongs to the class of convergence, it has a minimum type.

We consider that the functions A_1 and A_2 have the same category of growth if they have the same order, and in the case of the finite order, they have the same type and simultaneously belong to the class of convergence or divergence.

The function A_2 has a higher growth category than the function A_1 if one of the conditions is hold:

a) $\rho[A_2] > \rho[A_1]$;

b) $\rho[A_2] = \rho[A_1]$, but $T[A_2] = +\infty$ and $T[A_1] < +\infty$;

c) $\rho[A_2] = \rho[A_1]$, but $0 < T[A_2] \leq +\infty$ and $T[A_1] = 0$;

d) $\rho[A_2] = \rho[A_1], T[A_2] = T[A_1] = 0$ but A_1 belongs to the class of divergence, and A_2 belongs to the class of convergence.

Theorem 1.12. *Let* $A_1(r) \to +\infty \; (r_0 \leq r \to +\infty)$, *and*

$$A_2(r) = c_0 + \int_{r_0}^{r} \frac{A_1(t)}{t} dt, \quad c_0 = const.$$

Then the functions A_1 *and* A_2 *have the same category of growth.*

Proof. We can assume that $c_0 = 0$ and $r_0 = 1$. Since

$$A_2(er) \geq \int_r^{er} \frac{A(t)}{t} dt \geq A_1(r) \int_r^{er} \frac{dt}{t} = A_1(r) \,, \tag{1.26}$$

then $\rho[A_1] \leq \rho[A_2]$ and if $\rho[A_1] = \rho[A_2] = \rho$, then $T[A_1] \leq T[A_2] e^{\rho}$. On the other hand, if $\rho[A_1] \leq br^a, r \geq r_1$, then

$$A_2(r) \leq \int_{r_1}^{r} \frac{bt^a}{t} dt = K + \frac{b}{a} r^{\alpha}, \quad K = const,$$

that is $\rho[A_2] \leq \rho[A_1]$ and if $\rho[A_1] = \rho[A_2] = \rho$, then $T[A_2] \leq T[A_2] / \rho$.

It follows that the functions A_1 and A_2 have the same order ρ and if $0 < \rho < +\infty$, then they have the same type. In the case when $\rho = 0$, the function A_1 has the maximum type and, because of (1.26), the function A_2 has the same type.

Finally, if $0 < \rho < +\infty$,

$$\int_1^{\infty} \frac{A_2(t)}{t^{\rho+1}} dt = \int_1^{\infty} \frac{dt}{t^{\rho+1}} \int_1^{t} \frac{A_1(r)}{r} dr = \int_1^{\infty} \frac{A_1(r)}{r} dr \int_r^{\infty} \frac{dt}{t^{\rho+1}} = \frac{1}{\rho} \int_1^{\infty} \frac{A_1(r)}{r^{\rho+1}} dt \,,$$

i. e. the functions A_1 and A_2 belong to the class of convergence or divergence simultaneously.

Theorem 1.12 is proved.

Proximate order

The function $\rho(r)$ on $[0;+\infty)$ is called the proximate order if

1) $\rho(r)\geq 0, 0\leq r<+\infty$;
2) $\rho(r)\to\rho\in[0;+\infty), r\to+\infty$;
3) $\rho(r)$ is continuously differentiated function on $[0;+\infty)$;
4) $r\rho'(r)\ln r\to 0, r\to+\infty$

If the difference between the proximate orders is $o(1/\ln r)$, when $r\to+\infty$, then those two proximate orders are called equivalent. Note that for equivalent proximate orders $\rho_1(r)$ and $\rho_2(r)$

$$r^{\rho_1(r)-\rho_2(r)}=\exp\{(\rho_1(r)-\rho_2(r))\ln r\}=e^{o(1)}\to 1(r\to+\infty).$$

Note some of the simplest properties of the proximate order.

Lemma 1.6. *If $\rho(r)$ is the proximate order, then the function $l(r)=r^{\rho(r)-\rho}$ is the slowly varying.*

Indeed

$$\frac{rl'(r)}{l(r)}=\frac{d\ln l(r)}{d\ln r}=r\rho'(r)\ln r+\rho(r)-\rho\to 0\quad(r\to+\infty),$$

that is the function l is the slowly varying.

Lemma 1.7. *If $\rho(r)$ is the proximate order and $\rho>0$, then the function $V(r)=r^{\rho(r)}$ is increasing on $[r_0;+\infty)$.*

Indeed

$$V'(r)=(\rho(r)+r\rho'(r)\ln r)r^{\rho(r)-1}=(1+o(1))\rho r^{\rho(r)-1},\quad(r\to+\infty),$$

that is, $V'(r)>0$ for all enough large r.

Lemma 1.8. *If* $\rho(r)$ *is the proximate order, and* a *and* b *are the fixed numbers,* $0<a<b<+\infty$*, then for each* $\varepsilon>0$, $r\geq r_0(\varepsilon)$ *and each* $c\in[a;b]$

$$(1-\varepsilon)c^{\rho}r^{\rho(r)}\leq (cr)^{\rho(cr)}\leq (1+\varepsilon)c^{\rho}r^{\rho(r)}.$$

In fact, let $l(r)=r^{\rho(r)-\rho}$. Then $\dfrac{l(cr)}{l(r)}=\dfrac{(cr)^{\rho(cr)}}{c^{\rho}r^{\rho(r)}}$ and since by the Lemma 1.6 the function l is slowly varying, then by the Karamaty\`s Theorem $\dfrac{l(cr)}{l(r)}\to 1, r\to+\infty$ uniformly in relation to $c\in[a,b]$.

As can be seen from the Lemma 2.3, the function $r^{\rho(r)}$ behaves like a function r^{ρ}. This is also evidenced by the Lemma 2.4, the correctness of which is easily checked using the H'Lopital\`s rule.

Lema 1.9. *If* $\rho(r)$ *is the proximate order and* $\rho(r)\to\rho>0$*, then, when* $r\to+\infty$

$$\int_1^r t^{\rho(t)-q}dt=\frac{(1+o(1))}{\rho+1-q}r^{\rho(r)-q+1},\quad q<\rho+1,$$

and

$$\int_r^{\infty} t^{\rho(t)-q}dt=\frac{(1+o(1))}{q-\rho-1}r^{\rho(r)-q+1},\quad q>\rho+1.$$

Let's prove, for example, the first of these correlations. We have

$$\lim_{r\to+\infty}\frac{\int_1^r t^{\rho(t)-q}dt}{r^{\rho(r)-q+1}}=\lim_{r\to+\infty}\frac{r^{\rho(r)-q}}{r^{\rho(r)-q}\left(\rho(r)+1-q+r\rho'(r)\ln r\right)}=$$

$$\lim_{r\to+\infty}\frac{1}{\rho(r)+1-q+r\rho'(r)\ln r}=\frac{1}{\rho+1-q}.$$

Let A be a positive non-decreasing function on $[r_0;+\infty)$. The proximate order $\rho(r)$ is called the proximate order of the function A, if

$$\overline{\lim_{r\to+\infty}} \frac{A(r)}{r^{\rho(r)}} = \tau^* \in (0;+\infty). \tag{1.27}$$

The number τ^* is called the value of the type ot the function A relatively to the proximate order $\rho(r)$. Of course, if $\rho(r)$ is the proximate order of the function A and $\rho(r)\to\rho(r\to+\infty)$, then $\rho=\rho[A]$ is the order of the function A. Note that if in (1.27) $\rho(r)$ replace by the equivalent proximate order, then the value of the type τ^* will not change.

The proximate order $\lambda(r)$ is called the lower proximate order of the function A , as

$$\underline{\lim}_{r\to+\infty} \frac{A(r)}{r^{\lambda(r)}} = \tau_* \in (0;+\infty).$$

It is clear that $\lambda(r)\to\lambda=\lambda[A](r\to+\infty)$, where $\lambda[A]$ is the lower order of the function A.

The proof of the Valiron`s Theorem 1.13 belongs to S. Shah.

Theorem 1.13. *For each positive non-decreasing function A of the finite order, there exists the proximate order* $[0;+\infty)$.

Proof. It is enough to construct a function $\rho(r)$ that, instead of condition 3) in the definiton of the proximate order, satisfies the weaker condition, that $\rho(r)$ is continuous on $[0;+\infty)$ and continuously differentiated on $[0;+\infty)$ with exception of isolated points, in which there exist the one-sided derivatives. Because in the enough small neighborhoods of angular points $\rho(r)$ can be replaced by $o(1/\ln r)$, $r\to+\infty$ so that the new function is continuously differentiated and the conditions 1), 2), 4) and (1.27) are not violated. Note also, if $A(r)=O(1)$, $r\to+\infty$, then to prove the theorem, it is enough to choose $\rho(r)\equiv 0$.

So, let $A(r)\uparrow +\infty, r\to+\infty$, and ρ be the order of the function A. Let's denote

$$d(r)=\frac{\ln^{+}A(r)}{\ln r},$$

so $\overline{\lim\limits_{r\to+\infty}}\, d(r)=\rho$, there are two possible cases:

1) there exists a sequence $(r_k)\uparrow+\infty, k\to\infty$ such that $d(r_k)>\rho$;

2) the inequality $r\ge r_0$ is hold for all $d(r)\le\rho$.

Let's consider the first case. Let $\varphi(r)=\max\{d(x):x\ge r\}$. Then $\varphi(r)\downarrow\rho, r\to+\infty$ and the set $M=\max\{r:d(r)=\varphi(r)\}$ is unbounded. Let $r_1\ge \exp_3 1$ and $r_1\in M$. Let's define

$$\rho(r)=\varphi(r_1),\quad 0\le r\le r_1.$$

Denote $t_1=\min\{t\in N: t\ge r_1+1, \varphi(t)<\varphi(r_1)\}$ and

$$\rho(r)=\varphi(r_1),\quad r_1\le r\le t_1.$$

Let u_1 be the abscissa of the first point of curves`s intersection $y=\varphi(r)$ and $y=\rho(r_1)-\ln_3 r+\ln_3 t_1$ at $r\ge t_1$. Denote

$$\rho(r)=\rho(r_1)-\ln_3 r+\ln_3 t_1,\quad t_1\le r\le u_1.$$

It is obviously that $\rho(r)\ge\varphi(r)$ on $[t_1,u_1]$ and $\rho(u_1)=\varphi(u_1)$.

Finally, let $r_2=\min\{M\cap\{r\ge u_1\}\}$. If $r_2>u_1$, then

$$\rho(r)=\varphi(r)=\varphi(r_2),\quad u_1\le r\le r_2.$$

Next, with r_2 we make the same consideration as with r_1, and since

$$r_n-r_{n-1}\ge u_n-r_{n-1}\ge t_n-r_{n-1}\ge 1,$$

then, repeating this process, we define $\rho(r)$ on $[0;+\infty)$.

Thus, the constructed function $\rho(r)$ is continuous and has a continuous derivative everywhere, with the exception of points t_n and u_n. in which there exists one-sided derivatives. Since either $\rho'(r) \equiv 0$ or $\rho'(r) = -\dfrac{1}{r \ln r \ln_2 r}$, then the condition 4) is hold. By construction $\rho(r) \geq \varphi(r) \geq d(r)$ at $r \geq r_1$, $\rho(r_n) \geq \varphi(r_n) \geq d(r_n)$ for all n and $\rho(r) \to \rho, r \to +\infty$. Therefore, $\ln A(r) \leq d(r) \ln r \leq \rho(r) \ln r$ when $r \geq r_1$ and $\ln A(r_n) \leq d(r_n) \ln r_n \leq \rho(r_n) \ln r_n$ for everyone $n \in N$. It follows that

$$\overline{\lim_{r \to +\infty}} \frac{A(r)}{r^{\rho(r)}} = 1, \tag{2.6}$$

and the theorem in this case is proved.

Let's consider the second case, $0 \leq d(r) < \rho$ for all $r_0 \geq \exp_3 1$. Let $\psi(r) = \max\{d(x) : r_0 \leq x \leq r\}$. Then $\psi(r)$ □ $\rho, r \to +\infty$, and the set $L = \{r \geq r_0 : d(r) = \psi(r)\}$ is unbounded. Let $r_1 > r_0$, and s_1 is the largest of the abscissa of the points of curves\`s intersection $y = \psi(r)$ and $y = \rho + \ln_3 r - \ln_3 r_1$. Obviously, there exists the point s_1, when r_1 is large enough such $0 < s_1 < r_1$ and $L \cap [r_0, s_1] \neq \varnothing$. Let's denote $t_1 = \max\{L \cap [r_0, s_1]\}$ and

$$\rho(r) = \psi(t_1) = d(t_1), \quad 0 \leq r \leq s_1.$$

Take now $r_2 \geq r_1$ so large that the largest of the abscissa s_2 of the points of curves\`s intersection $y = \psi(r)$ and $y = \rho + \ln_3 r - \ln_3 r_2$ satisfy the conditions $r_1 < s_2 < r_2$ and $L \cap [r_1, s_2] \neq \varnothing$. Let's denote $t_2 = \max\{L \cap [r_1, s_2]\}$, and let u_1 is such a point from $[s_1, r_1]$, at which $\psi(r) = \rho + \ln_3 r - \ln_3 r_1 2$. Let\`s take

$$\rho(r) = \begin{cases} \rho + \ln_3 r - \ln_3 r_1, & s_1 \leq r \leq u_1, \\ \psi(t_2) = d(t_2), & u_1 \leq r \leq s_2. \end{cases}$$

Continuing this process, by inequality $r_n \geq r_{n-1} + 1$ we can construct $\rho(r)$ on $[0; +\infty)$.

From the construction it can be seen that the function $\rho(r)$ is continuous and has a continuous derivative everywhere, with the exception in points s_n and u_n, in which there exist one-sided derivatives $\rho'(r)\equiv 0$, or $\rho'(r)=\dfrac{1}{r\ln r\ln_2 r}$, that is, condition 4) takes place. It is clear, that $\rho(r)\to\rho(r\to+\infty)$, $\rho\ge\rho(r)\ge\psi(r)\ge d(r)$ when $r\ge r_0$ and $\rho(t_n)=\psi(t_n)=d(t_n)$ for all $n\in N$. Hence, it follows (2.6). Theorem 2.2 is proved.

Theorem 1.14. *For each positive non-decreasing function* A *on* $[0;+\infty)$ *of the finite lower order* λ *there exists its proximate lower order.*

Proof. Let

$$d_1(r)=\frac{\ln r}{\ln r+\ln^+ A(r)},\quad r\ge 2,$$

such that $\overline{\lim\limits_{r\to+\infty}}\, d_1(r)=\dfrac{1}{1+\lambda}$. Let's construct the proximate order $\rho_1(r)$ such that $\rho_1(r)\to\dfrac{1}{1+\lambda}$ when $r\to+\infty$, $\rho_1(r)\to\dfrac{1}{1+\lambda}$ when $r\to+\infty$, $0<\rho_1(r)<1$, $d_1(r)\le\rho_1(r)$ for all $r\ge r_0$ and $d_1(r_n)\le\rho_1(r_n)$ for some sequence (r_n), that approaching to the $+\infty$. Let's denote $\lambda(r)=\dfrac{1}{\rho_1(r)}-1$. It is easy to check, that $\lambda(r)$ has all the properties of the proximate order, $\lambda(r)\to\lambda\,(r\to+\infty)$, $\ln A(r)\ge\lambda(r)\ln r$, when $r\ge r_0$ and $\ln A(r_n)\ge\lambda(r_n)\ln r_n$ for all $n\ge 1$, that is $\lambda(r)$ is the proximate lower order of the function A.

Poy's peaks

Theorem 1.15. *Let the functions* $\varphi,\varphi_1,\varphi_2$ *be positive and continuous on* $[r_0;+\infty)$ *, such that* φ_2/φ_1 *is non-decreasing on* $[r_0;+\infty)$ *and*

$$\overline{\lim_{r\to+\infty}}\frac{\varphi(r)}{\varphi_2(r)}=+\infty,\quad \overline{\lim_{r\to+\infty}}\frac{\varphi(r)}{\varphi_2(r)}=0\ . \tag{1.29}$$

Then there exists a sequence $(r_n)\uparrow +\infty, n\to\infty$, *such that*

$$\varphi(r)\le\begin{cases}\dfrac{\varphi(r_n)}{\varphi_1(r_n)}\varphi_1(r), & r_0\le r\le r_n,\\ \dfrac{\varphi(r_n)}{\varphi_2(r_n)}\varphi_2(r), & r_n\le r<+\infty.\end{cases}\tag{1.30}$$

Proof. Let's assume that $r_0, r_1, \dots, r_n$ are already known and denote

$$M_n=\max\left\{\frac{\varphi(r)}{\varphi_1(r)}: r_0\le r\le r_n\right\}$$

(in particular $M_0=\varphi(r_0)/\varphi_1(r_0)$). From the first condition (1.29) follows the existence of a number

$$a=\min\left\{r\ge r_n+1: \frac{\varphi(r)}{\varphi_1(r)}=M_n\right\}.$$

Let $b\ge a$, such that

$$\frac{\varphi(b)}{\varphi_2(b)}=\max\left\{\frac{\varphi(r)}{\varphi_2(r)}: r\ge a\right\}.$$

Let's choose $r_{n+1}\in[a;b]$,so that

$$\frac{\varphi(r_{n+1})}{\varphi_1(r_{n+1})}=\max\left\{\frac{\varphi(r)}{\varphi_1(r)}: a\le r\le b\right\},\tag{1.31}$$

and show that for r_{n+1} (1.30) is hold.

In fact, if $r_0\le r\le a$, then

$$\frac{\varphi(r)}{\varphi_1(r)}\le M_n=\frac{\varphi(a)}{\varphi_1(a)}\le\frac{\varphi(r_{n+1})}{\varphi_1(r_{n+1})},$$

and if $a\le r\le r_{n-1}$, then by the construction

$$\frac{\varphi(r)}{\varphi_1(r)}\le\frac{\varphi(r_{n+1})}{\varphi_1(r_{n+1})}.$$

So, the first inequality (2.8) on $[r_0, r_{n+1}]$ with r_{n+1} instead of r_n is hold.

Further, since $\varphi_{2/}\varphi_1$ is non-decreasing on $[r_0;+\infty)$, then from (2.9) for $r_{n+1} \le r \le b$ we have

$$\frac{\varphi(r)}{\varphi_2(r)} = \frac{\varphi(r)\varphi_1(r)}{\varphi_1(r)\varphi_2(r)} \le \frac{\varphi(r_{n+1})\varphi_1(r_{n+1})}{\varphi_1(r_{n+1})\varphi_2(r_{n+1})} = \frac{\varphi(r_{n+1})}{\varphi_2(r_{n+1})},$$

and when $r \ge b$ the second inequality is hold

$$\frac{\varphi(r)}{\varphi_2(r)} = \frac{\varphi(b)}{\varphi_2(b)} \le \frac{\varphi(r_{n+1})}{\varphi_2(r_{n+1})}.$$

So, the second inequality (2.8) on $[r_{n+1};+\infty)$ with r_{n+1} instead r_n is hold. The Theorem 2.4 is proved.

The sequence (r_n) (the existence of which is already proved) is called the sequence of Poy's peak.

Here are the simplest corollaries of the Theorem 1.15.

Corollary 1.3. *Let* φ *be the positive continuous non-decreasing function on* $[r_0;+\infty)$ *of the order* $\rho \in (0;+\infty)$ *and* $\varepsilon > 0$ *is an arbitrary number. Then there exists the sequence* $(r_n) \uparrow +\infty, n \to \infty$ *of the Poy`s peaks, such that*

$$\varphi(r) \le \begin{cases} \varphi(r_n)\left(\dfrac{r}{r_n}\right)^{\rho-\varepsilon}, & r_0 \le r \le r_n, \\ \varphi(r_n)\left(\dfrac{r}{r_n}\right)^{\rho+\varepsilon}, & r_n \le r < +\infty. \end{cases} \tag{1.32}$$

Let's denote $\varphi_1(r) = r^{\rho-\varepsilon}$ and $\varphi_2(r) = r^{\rho+\varepsilon}$. Since φ has the order ρ, then

$$\frac{\varphi_2(r)}{\varphi_1(r)} = r^{2\varepsilon} \uparrow +\infty, \quad r \to +\infty,$$

$$\overline{\lim_{r\to+\infty}} \frac{\varphi(r)}{\varphi_1(r)} = \overline{\lim_{r\to+\infty}} \frac{\varphi(r)}{r^{\rho-\varepsilon}} = +\infty,$$

$$\overline{\lim_{r\to+\infty}} \frac{\varphi(r)}{\varphi_2(r)} = \overline{\lim_{r\to+\infty}} \frac{\varphi(r)}{r^{\rho+\varepsilon}} = 0.$$

Therefore, by the Theorem 1.15, there exists a sequence of the Poy`s peaks, for which (1.32) is hold.

Corollary 1.4. *Let* φ *be the positive continuous non-decreasing function on* $[r_0;+\infty)$ *of the lower order* $\lambda\in(0;+\infty)$*, and* $\varepsilon\in(0;\lambda)$ *is an arbitrary number. Then there exists the sequence* $(r_n)\uparrow+\infty,\ n\to\infty$ *of the Poy`s peaks, such that*

$$\varphi(r)\geq\begin{cases}\varphi(r_n)\left(\dfrac{r}{r_n}\right)^{\lambda+\varepsilon}, & r_0\leq r\leq r_n,\\[2ex] \varphi(r_n)\left(\dfrac{r}{r_n}\right)^{\lambda-\varepsilon}, & r_n\leq r<+\infty.\end{cases} \tag{1.33}$$

In fact, let's denote $\varphi_1(r)=\dfrac{\varphi^2(r)}{r^{\lambda+\varepsilon}}$ and $\varphi_2(r)=\dfrac{\varphi^2(r)}{r^{\lambda-\varepsilon}}$. Since φ has a lower order λ, then

$$\frac{\varphi_2(r)}{\varphi_1(r)}=r^{2\varepsilon}\uparrow+\infty,\quad r\to+\infty,$$

$$\overline{\lim_{r\to+\infty}}\frac{\varphi(r)}{\varphi_1(r)}=\overline{\lim_{r\to+\infty}}\frac{r^{\lambda+\varepsilon}}{\varphi(r)}=+\infty,$$

$$\overline{\lim_{r\to+\infty}}\frac{\varphi(r)}{\varphi_2(r)}=\overline{\lim_{r\to+\infty}}\frac{r^{\lambda-\varepsilon}}{\varphi(r)}=0.$$

Therefore, by the Theorem 1.15, there exists a sequence (r_n) of the Poy's peaks such that

$$\varphi(r)\leq\begin{cases}\dfrac{\varphi(r_n)r_n^{\lambda+\varepsilon}}{\varphi^2(r_n)}\cdot\dfrac{\varphi^2(r)}{r^{\lambda+\varepsilon}}, & r_0\leq r\leq r_n,\\[2ex] \dfrac{\varphi(r_n)r_n^{\lambda-\varepsilon}}{\varphi^2(r_n)}\cdot\dfrac{\varphi^2(r)}{r^{\lambda-\varepsilon}}, & r_n\leq r<+\infty,\end{cases}$$

whence the inequalities (1.33) follow.

If we use the proximate order, then we can get much more flexible statements.

Theorem 1.16. *Let φ be the positive continuous non-decreasing function on $[r_0;+\infty)$ of the order $\rho\in(0;+\infty)$. Then there exists the increasing sequences $(r_n),(\varepsilon_n)$ and the non-negative sequences $(a_n),(A_n)$ and (δ_n), such that*

1) $\varepsilon_n\to 0,\delta_n\to 0$ and $a_n\to 0$, when $n\to\infty$;

2) $A_n\to+\infty$ and $a_n r_n\to+\infty$, when $n\to\infty$;

3) for all $n\ge n_0$ and $r\in[a_n r_n, A_n r_n]$

$$\varphi(r)\le(1+\delta_n)\varphi(r_n)\left(\frac{r}{r_n}\right)^{\rho+\varepsilon_n} . \tag{1.34}$$

Proof. Let $\rho(r)$ be the proximate order of the function φ, which was constructed in the process of proving the Theorem 1.14 (Valiron`s theorem). Then

$$\overline{\lim_{r\to+\infty}}\, r^{-\rho(r)}\varphi(r)=1$$

and

$$|\rho'(r)|\le\frac{1}{r\ln r\ln_2 r}\quad (r\ge r_0).$$

Let's denote

$$\varphi_1(r)=r^{\rho(r)-1/\ln_2 r},\quad \varphi_2(r)=r^{\rho(r)+1/\ln_2 r}.$$

Then

$$\frac{\varphi_2}{\varphi_1}=r^{2/\ln_2 r}=\exp\left\{\frac{2\ln r}{\ln_2 r}\right\}\uparrow+\infty\quad (r_0\le r\to+\infty),$$

$$\overline{\lim_{r\to+\infty}}\frac{\varphi(r)}{\varphi_1(r)}=\overline{\lim_{r\to+\infty}}\frac{\varphi(r)}{r^{\rho(r)}}r^{1/\ln_2 r}=\overline{\lim_{r\to+\infty}}\left\{\frac{\ln r}{\ln_2 r}\right\}=+\infty,$$

$$\overline{\lim_{r\to+\infty}}\frac{\varphi(r)}{\varphi_2(r)}=\overline{\lim_{r\to+\infty}}\frac{\varphi(r)}{r^{\rho(r)}}r^{-1/\ln_2 r}=\overline{\lim_{r\to+\infty}}\left\{-\frac{\ln r}{\ln_2 r}\right\}=0.$$

Therefore, by the Theorem 1.15, there exists the increasing sequence (r_n) of the Poy's peaks, that is tendinging to $+\infty$, such that

$$\varphi(r) \le \begin{cases} \varphi(r_n) r^{\rho(r)-1/\ln_2 r} r_n^{-\rho(r_n)+1/\ln_2 r_n}, & r_0 \le r \le r_n, \\ \varphi(r_n) r^{\rho(r)+1/\ln_2 r} r_n^{-\rho(r_n)-1/\ln_2 r_n}, & r_n \le r < +\infty. \end{cases} \tag{1.35}$$

Denote

$$A_n = \sqrt{\ln_2 r}, \quad a_n = 1/A_n,$$

such tnat $A_n \to +\infty$, $a_n \to 0$ and $a_n r_n \to +\infty$, when $n \to \infty$.

If $a_n r_n \le r \le r_n$, then from the first inequality (2.13) we get

$$\varphi(r) \le \varphi(r_n) \left(\frac{r}{r_n} \right)^{\rho(r_n)-1/\ln_2 r_n} r^{\rho(r)-\rho(r_n)+1/\ln_2 r_n - 1/\ln_2 r} \le$$

$$\le \varphi(r_n) \left(\frac{r}{r_n} \right)^{\rho(r_n)-1/\ln_2 r_n} r^{\rho(r)-\rho(r_n)} =$$

$$= \varphi(r_n) \left(\frac{r}{r_n} \right)^{\rho+\rho(r_n)-\rho-1/\ln_2 r_n} \exp\left\{ \left(\rho(r)-\rho(r_n)\right)\ln r \right\}. \tag{1.36}$$

Let's denote $\varepsilon_n = \rho(r_n) - \rho - 1/\ln_2 r_n$, such that $\varepsilon_n \to 0 (n \to \infty)$. Then

$$\left| \left(\rho(r) - \rho(r_n)\right) \ln r \right| = \left| \int_{r_n}^{r} \rho'(\xi) d\varepsilon \right| \ln r \le \int_{r_n}^{r} \left|\rho'(\xi)\right| d\varepsilon \ln r \le$$

$$\int_{r_n}^{r} \frac{d\varepsilon}{\xi \ln \xi \ln_2 \xi} \le \frac{1}{\ln r \ln_2 r} \int_{r_n}^{r} \frac{d\varepsilon}{\xi} \ln r = \frac{\ln r_n - \ln r}{\ln_2 r} \le$$

$$\le \frac{\ln r_n - \ln(a_n r_n)}{\ln_2 r} = \frac{\ln A_n}{\ln(\ln r_n - \ln A_n)} =$$

$$= \frac{1}{2} \frac{\ln_3 r_n}{\ln\left(\ln r_n - \frac{1}{2} \ln_3 r_n \right)} = \frac{1+o(1)}{2} \frac{\ln_3 r_n}{\ln_2 r_n} \to 0 \quad (n \to \infty).$$

Therefore, from (1.36) it follows that

$$\varphi(r) \le \varphi(r_n) \left(\frac{r}{r_n} \right)^{\rho+\varepsilon_n} e^{o(1)} = (1+o(1)) \varphi(r_n) \left(\frac{r}{r_n} \right)^{\rho+\varepsilon_n}, \quad n \to \infty,$$

and there exists the sequence $(\delta_n)\to 0\,(n\to\infty)$, that the inequality (1.34) is hold.

If $r_n \le r \le A_n r_n$, then from the second inequality (1.35) we have

$$\varphi(r)\le\varphi(r_n)\left(\frac{r}{r_n}\right)^{\rho(r_n)+1/\ln_2 r_n} r^{\rho(r)-\rho(r_n)} r^{1/\ln_2 r_n-1/\ln_2 r}=$$

$$=\varphi(r_n)\left(\frac{r}{r_n}\right)^{\rho+\varepsilon_n}\exp\left\{\left(\rho(r)-\rho(r_n)\right)\ln r\right\},$$

where $\varepsilon_n=\rho(r_n)-\rho+1/\ln_2 r_n\to 0\,(n\to\infty)$, and

$$\left|\left(\rho(r)-\rho(r_n)\ln r\right)\right|\le\int_{r_n}^{r}\frac{d\varepsilon}{\xi\ln\xi\ln_2\xi}\ln r\le\frac{\ln r-\ln r_n}{\ln r_n\ln_2 r}\ln r\le$$

$$\le\frac{\ln(A_n r_n)-\ln r_n}{\ln r_n\ln_2 r_n}\ln(A_n r_n)=\frac{\ln A_n}{\ln r_n\ln_2 r_n}(\ln A_n+\ln r_n)=$$

$$=\frac{1}{2}\frac{\ln_3 r_n}{\ln r_n\ln_2 r_n}\left(\ln r_n+\frac{1}{2}\ln_3 r_n\right)=\frac{1+o(1)}{2}\frac{\ln_3 r_n}{\ln_2 r_n}\to 0(n\to\infty).$$

Hence, we again come to the existence of a sequence $(\delta_n)\to 0\,(n\to\infty)$, such that the inequality (1.34) takes place. The theorem is proved.

Theorem 1.17. *Let* φ *be the positive continuous non-decreasing on* $[r_0;+\infty)$ *function of the lower order* $\lambda\in(0;+\infty)$*. Then there exists the sequences* $(r_n),(\varepsilon_n),(a_n),(A_n)$ *and* (δ_n) *that satisfy the conditions 1) - 2) of the Theorem 1.16 and such that for all* $n\ge n_0$ *and* $r\in[a_n r_n;A_n r_n]$

$$\varphi(r)\ge(1-\delta_n)\varphi(r_n)\left(\frac{r}{r_n}\right)^{\lambda+\varepsilon_n}.$$

The proving of this theorem is similar to the proving of the Theorem 1.16, but we need to choose φ_1 and φ_2 by the next way:

$$\varphi_1(r)=\varphi^2(r)r^{-\lambda(r)-1/\ln_2 r},\quad \varphi_2(r)=\varphi^2(r)r^{-\lambda(r)-1/\ln_2 r}.$$

Borel-Nevanlinna lemma

Theorem 1.18. *Let* u *be the continuous function on* $[r_0;+\infty)$ *and* $u(r)\uparrow +\infty, r\to+\infty$, *and* φ *be the positive continuous function on* $[u_0;+\infty)$ $(u_0=u(r_0))$, *such that* $\varphi(r)\downarrow 0, r\to+\infty$, *and*

$$\int_{u_0}^{\infty}\varphi(u)du<+\infty \ . \tag{1.37}$$

Then for all $r\geq r_0$, except, perhaps, the set of the finite measure

$$u\big(r+\varphi(u(r))\big)<u(r)+1. \tag{1.38}$$

Proof. Let`s denote $E, E\subset[r_0;+\infty)$ as the closed set, on which (1.38) is not hold, that is

$$u\big(r+\varphi(u(r))\big)\geq u(r)+1. \tag{1.39}$$

Let $E_r=E\bigcap[r;+\infty)$. Suppose that for each $r\geq r_0$ the set $E_r\neq\varnothing$, otherwise the statement of the theorem is obvious.

Let

$$r_1=\min E_{r_0}=\min\{r\in E: r\geq r_0\},$$

$$r_1'=\min\{r: u(r)=u(r_1)+1\}.$$

Then $r_1'>r_1$ and $u\big(r_1+\varphi(u(r_1))\big)\geq u(r_1)+1=u(r_1')$. We obtain that $r_1+\varphi\big(u(r_1)\big)\geq r_1'$, i. e.

$$r_1'-r_1\leq\varphi\big(u(r_1)\big).$$

Suppose that $r_1,\dots,r_n$, $r_1',\dots,r_n'$ are already chosen and denote

$$r_{n+1}=\min E_{r_n'}=\min\{r\in E: r\geq r_n'\},$$

$$r_{n+1}'=\min\{r: u(r)=u(r_{n+1})+1\}.$$

Then we have

$$u\big(r_{n+1}+\varphi(u(r_{n+1}))\big)\geq u(r_{n+1})+1=u(r_{n+1}')$$

and

$$r'_{n+1} - r_{n+1} \le \varphi(u(r_{n+1})).$$

Hence, $u(r_n) \to +\infty$ and $r_n \to +\infty$, when $n \to \infty$.

From the definition of the sequence (r_n) follows also that on the intervals $(r'_{n-1}; r_n)$ (1.38) is hold. So, the set E, where (1.38) is not hold, is covered by the intervals $[r_n; r'_n]$. But

$$\sum_{n=1}^{\infty}(r'_n - r_n) \le \sum_{n=1}^{\infty}\varphi(u(r_n)) \le \sum_{n=1}^{\infty}\varphi(u_0 + n + 1) \le \varphi(u_0) + \int_{u_0}^{\infty}\varphi(u)\,du < +\infty,$$

and, consequently, the Theorem 1.18 is proved.

Remark. The condition (1.37) cannot be weakened. Indeed, let $\int_{u_0}^{\infty}\varphi(u)\,du = +\infty$. Denote $r(u) = \int_{u_0}^{u}\varphi(u)\,du,\, u \ge u_0$.

Then the reversed to $r(u)$ function $u(r)$ satisfies the conditions of the Theorem 1.18, but the equality (1.39) is hold on $[r_0; +\infty)$, because

$$u(r + \varphi(u(r))) - u(r) = \int_{r}^{r+\varphi(u(r))} u'(t)\,dt = \int_{r}^{r+\varphi(u(r))} \frac{dt}{r'(u(t))} =$$

$$= \int_{r}^{r+\varphi(u(r))} \frac{dt}{\varphi(u(t))} \ge \frac{1}{\varphi(u(r))}\int_{r}^{r+\varphi(u(r))} dt = 1.$$

Corollary 1.5. *If u is a positive function that satisfies the conditions of the Theorem 2.7 and $\varepsilon > 0$, then for all $r \ge r_0$, except, perhaps, the set of the finite measure,*

$$u\left(r + \frac{1}{\ln u(r)}\right) \le u(r)^{1+\varepsilon}. \tag{1.40}$$

In fact, the functions $u_1(r) = \sqrt{\ln u(r)}$ and $\varphi(u) = 1/u^2$ satisfy the conditions of the Theorem 2.7 and therefore, for all $r \ge r_0^*$, except, perhaps, the set of the finite measure $u_1(r + \varphi(u_1(r))) < u_1(r) + 1$, i. e.

$$\sqrt{\ln u\left(r+\left(\frac{1}{\sqrt{\ln u(r)}}\right)^2\right)}<\sqrt{\ln u(r)}+1.$$

Therefore, when $r \ge r_0 \ge r_0^*$

$$\ln u\left(r+\frac{1}{\ln u(r)}\right)\le \ln u(r)+2\sqrt{\ln u(r)}+1<(1+\varepsilon)\ln u(r).$$

The set *E* is called the set of the finite logarithmic measure, if

$$\int\limits_{E\cap[1;+\infty)} d\ln r<+\infty.$$

Corollary 1.6. *If a positive function* u *satisfies the conditions of the Theorem 1.18 and* $\varepsilon>0$*, then for all* $r\ge r_0$*, except, perhaps,s the set of the finite logarithmic measure*

$$u\left(r+\frac{1}{\ln u(r)}\right)\le u(r)^{1+\varepsilon} \quad . \tag{1.41}$$

In fact, if we apply the Corollary 1 to the function $u\left(e^r\right)$, we will get

$$u\left(\exp\left\{r+\frac{1}{\ln u\left(e^r\right)}\right\}\right)\le u\left(e^r\right)^{1+\varepsilon} \tag{1.42}$$

for all $r\ge r_0^*$, except, perhaps, the set of the finite measure. Let's denote $x=e^r$, that is $r=\ln x$. With such the replacement, the set of the finite measure passes into the set of the finite logarithmic measure and from (1.42) we receive

$$u\left(\exp\left\{\ln x+\frac{1}{\ln u(x)}\right\}\right)\le u(x)^{1+\varepsilon},$$

so, $u\left(x\exp\left\{\frac{1}{\ln u(x)}\right\}\right)\le u(x)^{1+\varepsilon}$. Since $\exp\left\{\frac{1}{\ln u(x)}\right\}\le 1+\frac{1}{\ln u(x)}$, then from the last correlation we get (1.41).

Counting function and convergence indicator

Let (z_n) be the complex sequence with a single cluster point in ∞, ordered so that $|z_n| \le |z_{n+1}|$ (among the members of this sequence can be the same members and also equal to zero). The function

$$n(r) = \sum_{|z_n| \le r} 1$$

is called the counting function of the sequence (z_n). It is the number of the members of the sequence (z_n) that are contained in a closed circle $\{z : |z| \le r\}$. The function

$$N(r) = \int_0^r \frac{n(t) - n(0)}{t} dt + n(0) \ln r$$

is called the average counting function or Nevanllina`s counting function of the sequence (z_n).

Let $z_1 = z_2 = \dots = z_m = 0$ and $z_{m+1} \ne 0$, and $r_0 \in (0; |z_{m+1}|)$ be an arbitrary number. Then, when $r \ge r_0$

$$N(r) = \int_0^{r_0} \frac{n(t) - m}{t} dt + \int_{r_0}^r \frac{n(t)}{t} dt - m \int_{r_0}^r \frac{dt}{t} + m \ln r = \int_{r_0}^r \frac{n(t)}{t} dt + m \ln r_0 . \qquad (1.43)$$

Therefore, by the Theorem 1.23, the functions $n(r)$ and $N(r)$ have the same growth category. However, the function $N(r)$ has advantages upon the function $n(r)$. It is continuous and continuously differentiated on $[0; +\infty)$, except the countable number of points, which have the one-sided derivatives. At the points, where the derivative exists, the inequality $\frac{dN(r)}{dr} \le \frac{n(r)}{r}$ is hold, that is $\frac{dN(r)}{dr} \le \frac{n(r)}{r}$. From (1.43) also follows that $N(r)$ is a logarithmically convex function.

Suppose that $|z_n|>0$ for all n, and $k(r)$ is the largest of the integers, such that $\left|z_{k(r)}\right| = \max\{|z_n| : |z_n| \le r\}$. Then, from (1.43) we get

$$N(r)=\int_0^r \frac{n(t)}{t}dt = \sum_{j=1}^{k(r)-1}\int_{|z_j|}^{|z_{j+1}|}\frac{n(t)}{t}dt+\int_{|z_k(r)|}^{|r|}\frac{n(t)}{t}dt=$$

$$=\sum_{j=1}^{k(r)-1} j\left(\ln\left|z_{j+1}\right|-\ln\left|z_j\right|\right)+k(r)\left(\ln r-\ln\left|z_{k(r)}\right|\right)=$$

$$=\ln|z_2|-\ln|z_1|+2\left(\ln|z_3|-\ln|z_2|\right)+\ldots+$$

$$+\left(k(r)-1\right)\left(\ln\left|z_{k(r)}\right|-\ln\left|z_{k(r)-1}\right|\right)+k(r)\left(\ln r-\ln\left|z_{k(r)}\right|\right)=$$

$$=-\ln|z_1|-\ln|z_2|-\cdots-\ln\left|z_{k(r)-1}\right|-\ln\left|z_{k(r)}\right|+k(r)\ln r=\sum_{j=1}^{k(r)}\ln\frac{r}{\left|z_j\right|}=\sum_{|z_k|\le r}\ln\frac{r}{\left|z_j\right|}. \quad (1.44)$$

Since the counting function of the complex sequence (z_n) is the counting function of the non-negative sequence $(\lambda_n)=(|z_n|)$, and the rejection of the finite number of the members of the sequence does not affect on the asymptotics of the counting function, then we will study only the positive sequences.

So, let $0<\lambda_n \uparrow +\infty, n\to\infty$. Suppose that

$$\sum_{n=1}^{\infty}\frac{1}{\lambda_n^t}<+\infty \quad (1.45)$$

for some $t\in(0;+\infty)$. Then the number

$$\tau=\inf\left\{t>0:\sum_{n=1}^{\infty}\frac{1}{\lambda_n^t}<+\infty\right\} \quad (1.46)$$

is called the convergence indicator of the sequence (λ_n). If $\sum_{n=1}^{\infty}\lambda_n^{-t}=+\infty$ for each $t>0$, then $\tau=+\infty$. Note that in the case, when $0<\tau<+\infty$, the series $\sum_{n=1}^{\infty}\lambda_n^{-\tau}$ can be both convergent and divergent.

Theorem 1.19. *Let* $0<t<+\infty$. *Then the condition (1.45) is equivalent to the condition*

$$\int_0^{\infty}\frac{n(r)}{r^{t+1}}dr<+\infty. \tag{1.47}$$

Proof. Let $0<r_0<\lambda_1$ and $r>r_0$. Then $n(t)\equiv 0$ on $[0;r_0]$. Taking the Stieltjes integral and integrating it by parts, we get

$$\sum_{\lambda_n\le r}\frac{1}{\lambda_n^t}=\int_{r_0}^{r}\frac{dn(x)}{x^t}=\frac{n(r)}{r^t}+t\int_{r_0}^{r}\frac{n(x)}{x^{t+1}}dx=\frac{n(r)}{r^t}+t\int_0^{r}\frac{n(x)}{x^{t+1}}dx. \tag{1.48}$$

It follows that

$$\int_0^{r}\frac{n(x)}{x^{t+1}}dx\le\frac{1}{t}\sum_{\lambda_n\le r}\frac{1}{\lambda_n^t}.$$

Approaching $r\to+\infty$, we get (1.47).

On the contrary, for all $\varepsilon>0$ and all $r\ge r^*(\varepsilon)$

$$\sum_{n=1}^{\infty}\frac{1}{\lambda_n^t}=t\int_0^{r}\frac{n(x)}{x^{t+1}}dx,$$

so that from (1.47) follows (1.45). Theorem 1.43 is proved.

From this theorem it follows that the convergence of the sequence can be denoteded by the equality

$$\tau=\inf\left\{t>0:\int_0^{\infty}\frac{n(x)}{x^{t+1}}dx<+\infty\right\}. \tag{1.49}$$

On the other hand, if $\rho=\varlimsup\limits_{r\to+\infty}\dfrac{\ln n(r)}{\ln r}$ is the order of the function $n(r)$, then, as had shown in 1.12,

$$\rho=\inf\left\{\mu>0:\int_0^{\infty}\frac{n(x)}{x^{\mu+1}}dx<+\infty\right\}. \tag{1.50}$$

From the formulas (1.49) and (1.50) follow the next theorem.

Theorem 1.20. *The convergence indicator of the sequence is equal to the order of its counting function.*

Density of sequence

Let $0<\lambda_n \uparrow +\infty, n\to\infty$, and $n(r)$ is the counting function of the sequence (λ_n). The values

$$D=\varlimsup_{r\to+\infty}\frac{n(r)}{r},\quad d=\varliminf_{r\to+\infty}\frac{n(r)}{r}$$

are called the upper and the lower densities of the sequence (λ_n). If $D=d=\Delta$, then the sequence is called the measurable one and Δ is called its density.

Let $r>0$ be an arbitrary number. If $\lambda_{n-1}<r<\lambda_n$, then $\dfrac{n-1}{\lambda_n}\le\dfrac{n(r)}{r}\le\dfrac{n}{\lambda_n}$ and if $r=\lambda_n=\lambda_{n+1}=\dots=\lambda_{n+q}<\lambda_{n+q+1}$, then $\dfrac{n(r)}{r}=\dfrac{n+q}{\lambda_{n+q}}$. Therefore,

$$D=\varlimsup_{r\to+\infty}\frac{n}{\lambda_n},\quad d=\varliminf_{r\to+\infty}\frac{n}{\lambda_n}.$$

The upper D^* and lower d^* average densities of the sequence (λ_n) we denote by the equalities

$$D^*=\varlimsup_{r\to+\infty}\frac{N(r)}{r},\quad d^*=\varliminf_{r\to+\infty}\frac{N(r)}{r},\quad N(r)=\int_0^r\frac{n(x)}{x}dx.$$

Theorem 1.21. The *inequalities* $D^* \le D \le eD^*$ *are hold.*

Proof. If $D < +\infty$, then $n(x) \le (D+\varepsilon)x$ for each $\varepsilon > 0$ and all $x \ge x_0 = x_0(\varepsilon)$. Then

$$N(r) \le \int_0^{x_0} \frac{n(x)}{x} dx + \int_{x_0}^{r} (D+\varepsilon) dx = (D+\varepsilon) r + const,$$

where $D^* \le D + \varepsilon$, ε is an arbitrary number. Therefore, we obtain the inequality $D^* \le D$, which is obvious if $D = +\infty$.

Then

$$N(r) \ge \int_{r/e}^{r} \frac{n(x)}{x} dx \ge n\left(\frac{r}{e}\right)$$

and

$$D^* \ge \overline{\lim_{r \to +\infty}} \frac{n(r/e)}{r} = \overline{\lim_{r \to +\infty}} \frac{n(r/e)}{e(r/e)} = \frac{D}{e},$$

that is, we have the inequality $D \le eD^*$. Theorem 1.21. proved.

Notice that the obtained estimates in Theorem 1.21 are exact. For example, if $\lambda_n = n$, then $n(r) = r + O(1)$ and $N(r) = r + O(\ln r)$ when $r \to +\infty$. Therefore, $D = D^* = 1$. Let`s prove that estimate $D \le eD^*$ is exact. For this it will be enough to construct the sequence (λ_n), such that at $D = 1$ and $D^* = 1/e$.

Let (ν_k) be the increasing sequence to $+\infty$ of natural numbers, $\nu_0 = 0$ and for $\nu_{k-1} < n \le \nu_k$ we denote

$$\lambda_n = \nu_k + \frac{n - \nu_{k-1}}{\nu_k - \nu_{k-1}}.$$

If $r > 0$ and $\lambda_n \le r < \lambda_{n+1}$, then

$$\frac{n(r)}{r} \le \frac{n}{\lambda_n} \le \frac{\nu_k}{\nu_k + \dfrac{n - \nu_{k-1}}{\nu_k - \nu_{k-1}}} \le 1,$$

and since $\lambda_{\nu_k} = \nu_k + 1$, then

$$1 \geq D = \overline{\lim_{n\to+\infty}} \frac{n}{\lambda_n} \geq \overline{\lim_{k\to+\infty}} \frac{\nu_k}{\lambda_{\nu_k}} = \overline{\lim_{k\to+\infty}} \frac{\nu_k}{\nu_k+1} = 1,$$

i.e. $D = 1$.

Since

$$\lambda_{\nu_{k+1}} = \nu_{k+1} + \frac{1}{\nu_{k+1} - \nu_k},$$

then, when $\nu_k \leq r \leq \nu_{k+1}$ the inequality $n(r) \leq \nu_k$ takes place. But

$$N(r) = \sum_{l\leq k-1} \int_{\nu_l}^{\nu_{l+1}} \frac{n(t)}{t} dt + \int_{\nu_k}^{r} \frac{n(t)}{t} dt, \quad \nu_k \leq r < \nu_{k+1}.$$

Therefore, when $\nu_k \leq r \leq \nu_{k+1}$, we obtain

$$\frac{N(r)}{r} \leq \frac{1}{\nu_k}\left\{\nu_1 + \sum_{1\leq k-1} \nu_l \ln\frac{\nu_{l+1}}{\nu_l}\right\} + \frac{\nu_k}{r}\ln\frac{r}{\nu_k}.$$

Since $\frac{1}{x}\ln x \leq \frac{1}{e}$ for all $x > 0$, then from the last inequality we get

$$\frac{N(r)}{r} \leq \frac{1}{\nu_k}\left\{\nu_1 + \sum_{1\leq k-1} \nu_l \ln\frac{\nu_{l+1}}{\nu_l}\right\} + \frac{1}{e}.$$

Until now, the sequence (ν_k) has been arbitrary. If we choose it so that $\nu_k = [\exp_2 k]$, then when $\nu_k \leq r < \nu_{k+1}$ we'll get

$$\frac{N(r)}{r} \leq \frac{\nu_1}{\nu_k} + \frac{\nu_{k-1}}{\nu_k}\ln\frac{\nu_k}{\nu_1} + \frac{1}{e} =$$

$$= \frac{1}{e} + o(1) + \frac{\exp_2(k-1) + O(1)}{\exp_2 k + O(1)} \ln\frac{\exp_2 k + O(1)}{\nu_1} =$$

$$= \frac{1}{e} + o(1) + (1 + o(1))e^k \frac{\exp_2(k-1)}{\exp_2 k} = \frac{1}{e} + o(1), \quad k \to \infty.$$

It follows that $D^* \leq 1/e$ and together with the inequality $D \leq eD^*$ we obtain the equality $D^* = 1/e$.

Theorem 1.22. *The inequalities* $d \le d^* \le d + d\ln\frac{D}{d}$ *are hold, where the expression on the right part should be considered as such that equals to the* $+\infty$, *when* $D = +\infty$ *and for arbitrary* d , *and equals to* 0, *when* $D < +\infty$ *and* $d = 0$.

Proof. If $d > 0$, then $n(x) \ge (d - \varepsilon)x$ for each $\varepsilon \in (0;d)$ and all $x \ge x_0(\varepsilon)$. Then

$$N(r) \ge \int_0^{x_0} \frac{n(x)}{x} dx + \int_{x_0}^{r} (d - \varepsilon) dx = (d - \varepsilon) r + const,$$

and we get the inequality $d^* \ge d - \varepsilon$ and, given the arbitrariness of ε, we have the inequality $d^* \ge d$, which is obvious, when $d = 0$.

Suppose that $D < +\infty$. Then $n(x) \le (D + \varepsilon)x$ for each $\varepsilon > 0$ and all $x \ge x_0(\varepsilon)$. Therefore, if $\alpha > 1$, then

$$N(\alpha r) = \int_0^{x_0} \frac{n(x)}{x} dx + \int_{x_0}^{r} \frac{n(x)}{x} dx + \int_{r}^{\alpha r} \frac{n(x)}{x} dx \le const + (D + \varepsilon) \int_{x_0}^{r} dx + \int_{r}^{\alpha r} \frac{dx}{x} =$$
$$= n(\alpha r)\ln\alpha + (D + \varepsilon) r + const,$$

that is

$$\frac{N(\alpha r)}{\alpha r} \le \frac{n(\alpha r)}{\alpha r}\ln\alpha + \frac{(D + \varepsilon)}{\alpha} + o(1), \quad r \to +\infty.$$

It follows that

$$d^* \le \frac{D}{\alpha} + d\ln\alpha \tag{1.51}$$

for each $\alpha > 1$. If $D = +\infty$, then the inequality (1.51) is obvious and in this case $d^* \le +\infty$. If $D < +\infty$ and $d = 0$, then $d^* \le D/\alpha$ and approaching $\alpha \to +\infty$ we get $d^* = 0$.

Finally, let $0 < d \le D < +\infty$. Denote $\psi(\alpha) = \frac{D}{\alpha} + d\ln\alpha$. Then $\psi(1) = D, \psi(+\infty) = +\infty$, the equation $\psi'(\alpha) = 0$ has the unique solution $\alpha^* = D/d$ and $\psi(\alpha^*) = d + d\ln\frac{D}{d} < D$. Hence, from the inequality (1.51) we have

$$d^* \le \min\left\{\frac{D}{\alpha} + d\ln\alpha : \alpha > 1\right\} = d + d\ln\frac{D}{d}.$$

Theorem 1.22 is proved.

Let again $0 < \lambda_n \uparrow +\infty, n \to \infty, n(r)$ is the counting function of the sequence (λ_n) and $\xi \in [0;1)$. The values

$$D(\xi) = \overline{\lim_{r \to +\infty}} \frac{n(r) - n(\xi r)}{r - r\xi}, \quad d(\xi) == \underline{\lim_{r \to +\infty}} \frac{n(r) - n(\xi r)}{r - r\xi}$$

are called the upper and lower densities of the sequence (λ_n) by the basis $\xi \in [0;1)$ correspondingly. It is clear that $D(0)=D$ and $d(0)=d$.

Theorem 1.23. *If*

$$\lambda_1 \geq h > 0, \quad \lambda_{n+1}-\lambda_n \geq h > 0 \quad (n \geq 1) \ , \tag{1.52}$$

then the functions $D(\xi)$ *and* $d(\xi)$ *are continuous on* $[0;1)$ *and*

$$0 \leq d(\xi) \leq d(0) \leq D(0) \leq D(\xi) \leq 1/h \, . \tag{1.53}$$

Proof. The inequalities $d(\xi) \geq 0$ and $d(0) \leq D(0)$ are obvious. Further, from the condition (1.52) it follows that the number of the members of the sequence (λ_n) , located on the interval $[a;b)$, does not exceed the number $(b-a)/h$. Consequently, for any $0 \leq \xi < \eta < 1$, we have

$$\overline{\lim_{t \to +\infty}} \frac{n(t\eta)-n(t\xi)}{r\eta - r\xi} \leq \frac{1}{h} . \tag{1.54}$$

When $\eta = 1$ it follows the inequality $D(\xi) \leq 1/h$. Finally, from the identity

$$\frac{n(t)}{t} = (1-\xi)\frac{n(t)-n(t\xi)}{t-t\xi} + \xi \frac{n(t\xi)}{t\xi}$$

we obtain the inequalities

$$d(0) \geq (1-\xi)d(\xi)+\xi d(0), \quad D(0) \geq (1-\xi)D(\xi)+\xi D(0)$$

and therefore, $d(\xi) \leq d(0) \leq D(0) \leq D(\xi)$. Hence, all inequalities (1.53) are proved. We only must to prove the continuity of the functions $d(\xi)$ and $D(\xi)$.

Let $0 \leq \xi < \eta < 1$. Using (1.54) and

$$(1-\xi)\frac{n(t)-n(t\xi)}{t-t\xi} = (1-\eta)\frac{n(t)-n(t\eta)}{t-t\eta} + \frac{n(t\eta)-n(t\xi)}{t},$$

we have

$$(1-\eta)d(\eta) \leq (1-\xi)d(\xi) \leq (1-\eta)d(\eta) + \frac{\eta-\xi}{h},$$

$$(1-\eta)D(\eta) \leq (1-\xi)D(\xi) \leq (1-\eta)D(\eta) + \frac{\eta-\xi}{h},$$

i.e.

$$-\frac{1}{h} \leq \frac{(1-\eta)d(\eta)-(1-\xi)d(\xi)}{\eta-\xi} \leq 0,$$

$$-\frac{1}{h} \le \frac{(1-\eta)D(\eta)-(1-\xi)D(\xi)}{\eta-\xi} \le 0.$$

From these inequalities follows the continuity of the functions $(1-\xi)d(\xi)$ and $(1-\xi)D(\xi)$, and hence, the continuity of $d(\xi)$ and $D(\xi)$. Theorem 1.23 is proved.

Corollary 1.7. *If* $D(\eta)=d(\xi)$ *for some* $0\le\xi,\eta<1$, *then the sequence* (λ_n) *is measurable. On the contrary, if the sequence* (λ_n) *is measurable, then for all* $0\le\xi,\eta<1$.

In fact, if $D(\eta)=d(\xi)$ for some $0\le\xi,\eta<1$, it follows from (1.53) that $D(0)=d(0)$, that is, the sequence is (λ_n) measurable. On the contrary, from the identity

$$\frac{n(t)-n(t\xi)}{t-t\xi} = \frac{1}{1-\xi}\frac{n(t)}{t} - \frac{\xi}{1-\xi}\frac{n(t\xi)}{t\xi}$$

and the existence of the limit $\lim\limits_{t\to+\infty}\frac{n(t)}{t}=\Delta$, it follows that

$$\lim_{t\to+\infty}\frac{n(t)-n(t\xi)}{t-t\xi} = \frac{\Delta}{1-\xi} - \frac{\Delta\xi}{1-\xi} = \Delta.$$

Theorem 1. 24. *There exists the limits*

$$\lim_{\xi\uparrow 1} d(\xi) = d(1), \quad \lim_{\xi\uparrow 1} D(\xi) = d(1)$$

and the inequalities are hold

$$d(1)\le d(\xi)\le d(0)\le D(0)\le D(\xi)\le D(1).$$

Proof. From the identity

$$\frac{1}{1-\xi^n}\sum_{\nu=1}^{n}\frac{n\left(t\xi^{\nu-1}\right)-n\left(t\xi^{\nu}\right)}{t\xi^{\nu-1}-t\xi^{\nu}}\left(\xi^{\nu-1}-\xi^{\nu}\right) = \frac{n(t)-n\left(t\xi^n\right)}{t-t\xi^n}$$

it follows that

$$D\left(\xi^n\right)\le\frac{1}{1-\xi^n}\sum_{\nu=1}^{n}D(\xi)\left(\xi^{\nu-1}-\xi^{\nu}\right)=D(\xi)$$

and, likewise, $\left(\xi^n\right) \le \delta(\xi)$. We will consider only the proving of the existence of the limit $\lim\limits_{\xi \uparrow 1} d(\xi) = d(1)$ and proving the inequality $d(1) \le d(\xi)$.

Let $m = \inf\{d(\xi) : 0 \le \xi < 1\}$. Then $d(\xi) \ge m, 0 \le \xi < 1$ and for each $\varepsilon \in (0;1)$ there exists $\gamma \in (0;1)$, such that $d(\gamma) < m + \varepsilon$. We'll choose ξ so that $1 - \varepsilon(1-\gamma) < \xi < 1$. Then $\xi > \gamma$ and there exists the natural number $n > 1$, such that $\xi^n \le \gamma < \xi^{n-1} < \xi$. Since the function $(1-\xi)d(\xi)$ is non-increasing, then we get

$$(1-\gamma)d(\gamma) \ge \left(1-\xi^{n-1}\right)d\left(\xi^{n-1}\right) \ge \left(1-\xi^{n-1}\right)d(\xi),$$

i.e.

$$d(\xi) \le d(\gamma)\frac{1-\gamma}{1-\xi^{n-1}} = d(\gamma)\left\{1 + \frac{\xi^{n-1}-\gamma}{1-\xi^{n-1}}\right\}.$$

But

$$\xi^{n-1} - \gamma \le \xi^n(1-\xi) < \varepsilon(1-\gamma) \le \varepsilon\left(1-\xi^n\right)$$

and

$$\frac{\xi^{n-1}-\gamma}{1-\xi^{n-1}} < \varepsilon\frac{1-\xi^n}{1-\xi^{n-1}} < \varepsilon\frac{n}{n-1} \le 2\varepsilon.$$

Therefore,

$$m \le d(\xi) \le d(\gamma)(1+2\varepsilon) < (m+\varepsilon)(1+2\varepsilon),$$

whence, due to the arbitrariness of ε, the existence of the limit $\lim\limits_{\xi \uparrow 1} d(\xi) = m = d(1) \le d(\xi)$ follows. Theorem 1.24 is proved.

The values $d(1)$ and $D(1)$ are called the minimal and the maximum densities of the sequence (λ_n) correspondingly.

Control questions

1. Formulate the definition of the upper and lower limits.
2. Give the examples of the convex functions.
3. Formulate the definition of a slowly variying function and give the example.
4. Formulate the definition of a semi-continuous function and give the example.
5. Define the class of convergence or divergence of the function.
6. Formulate the definition of the proximate order of the function and give the example.
7. Define the counting function and the convergence indicator of the sequence.
8. Formulate the definition of the upper and the lower densities of the sequence.

Examples of problem solving

Example. Find the upper and the lower limits of the functions:

a) $f(x)=\sin^2\frac{1}{x}+\frac{2}{\pi}arctg\frac{1}{x},\ x\to 0$;

b) $g(x)=\left(2-x^2\right)\cos\frac{1}{x},\ x\to 0$.

Solving.

a) Since $\inf\left\{\sin^2\frac{1}{x}\right\}=0$, when $x=x_n=-\frac{1}{n\pi}\left(n=1,2,\ldots\right)$

and

$$\lim_{n\to\infty}\frac{2}{\pi}arctg\frac{1}{x_n}=\inf\left\{\frac{2}{\pi}arctg\frac{1}{x}\right\}=-1,$$

then $\varliminf_{x\to 0} f(x)=\lim_{n\to\infty}\left(\sin^2 n\pi+\frac{2}{\pi}arctg\left(-n\pi\right)\right)=-1$. Similarly, $\sup\left\{\sin^2\frac{1}{x}\right\}=1$,

when $x=x_n=\frac{2}{\pi\left(1+2n\right)}\left(n=1,2,\ldots\right)$ and $\lim_{n\to\infty}\frac{2}{\pi}arctg\frac{1}{x_n}=\sup\left\{\frac{2}{\pi}arctg\frac{1}{x}\right\}=1$,

then

$$\varlimsup_{x\to 0} f(x)=\lim_{n\to\infty}\left(\sin^2\frac{\pi\left(1+2n\right)}{2}+\frac{2}{\pi}arctg\frac{\pi\left(1+2n\right)}{n}\right)=1+1=2\,.$$

b) Since $\inf\left\{\cos\frac{1}{x}\right\}=-1$, when $x=x_n=-\frac{1}{(2n-1)\pi}\,(n=1,2,\ldots)$ and $\lim\limits_{n\to\infty}\cos\frac{1}{x_n}=-1$, and $\lim\limits_{x\to 0}\left(2-x^2\right)=2$, then $\varliminf\limits_{x\to 0} g(x)=\lim\limits_{n\to\infty}\left(2-\frac{1}{(2n-1)^2\pi^2}\right)\cos(2n-1)\pi=-2$.

Similarly, $\varlimsup\limits_{x\to 0} g(x)=2$.

Tasks for independent work

Find the upper and the lower limits of the function:

1) $f(x)=x^2\sin x,\quad x\to 0;$

2) $f(x)=e^{\cos\left(1/x^2\right)},\quad x\to 0;$

3) $f(x)=arctg\,\frac{1}{x},\quad x\to 0;$

4) $f(x)=\left(\frac{1+x}{1+2x}\right)^x,\quad x\to\infty;$

5) $f(x)=2^x,\quad x\to\infty;$

6) $f(x)=\frac{\cos x}{x},\quad x\to 0;$

7) $f(x)=\frac{\sin x}{x},\quad x\to\frac{\pi}{2};$

8) $f(x)=e^{-x^2},\quad x\to\infty;$

9) $f(x)=\frac{1}{1+e^{1/x}},\quad x\to 0;$

10) $f(x)=e^{\cos\left(1/x^2\right)},\quad x\to 0;$

11) $f(x)=\sin\frac{1}{x},\quad x\to\frac{4}{\pi}.$

Capter II. Entire functions

Maximum modulus

The function $f(z), z = x + iy$ is called analytic at the point z_0, if it has a continuous derivative in some area of this point. If the function $f(z)$ is analytic on all complex plane, then $f(z)$ is called an entire function. From Taylor`s theorem we can get

$$f(z) = \sum_{n=0}^{\infty} a_n z^n , \tag{2.1}$$

convergence radius is $R = 1/\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|a_n|} = \infty$. Therefore, the condition

$$\overline{\lim_{n\to\infty}}\sqrt[n]{|a_n|} = 0$$

is hold.

The value $M(r) = \max\limits_{|z|\le r}\{|f(z)|\}, 0 \le r < \infty$, the growth rate of which is one of the most important characteristics of the entire function $f(z)$, when $r \to \infty$, is called the maximum modulus in the circle of the radius r.

Theorem 2.1 (Cauchy inequality). *For any* $n \in N$ *takes place the inequality*

$$|a_n r^n| \le M(r).$$

Proof. When $0 < r < R$ and $z = re^{i\varphi}$, we have

$$|f(z)|^2 = \left|f\left(re^{i\varphi}\right)\right|^2 = f\left(re^{i\varphi}\right)\overline{f\left(re^{i\varphi}\right)} =$$

$$= \sum_{n=0}^{\infty} a_n r^n e^{in\varphi} \sum_{k=0}^{\infty} \bar{a}_k r^k e^{-ik\varphi} = \sum_{n,k=0}^{\infty} a_n \bar{a}_k r^{n+k} e^{i(n+k)\varphi} ,$$

and the last series is uniformly convergent on $0 \le \varphi \le 2\pi$, because it is majored by the series

$$\sum_{n,k=0}^{\infty} |a_n||a_k| r^{n+k} = \left(\sum_{n=0}^{\infty} |a_n| r^n \right)^2 < \infty .$$

By integrating term by term , we will have

$$\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\varphi}\right) \right|^2 d\varphi = \sum_{n,k=0}^{\infty} a_n \bar{a}_k r^{n+k} \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-k)\varphi} d\varphi .$$

Since

$$\frac{1}{2\pi} \int_0^{2\pi} e^{i(n-k)\varphi} d\varphi = \begin{cases} 0, & n \neq k, \\ 1, & n = k, \end{cases}$$

then

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\varphi}\right) \right|^2 d\varphi \leq \{M(r)\}^2 ,$$

and Cauchy's inequality is an obvious consequence of the latest inequality.

Theorem 2.2 (Liouville’s). If for entire function $f(z)$ the condition

$$M(r) < Kr^q, \quad r > r_0, \quad K = const , \tag{2.2}$$

is hold, then $f(z)$ is a polynomial.

Indeed, from Cauchy’s inequality, when the condition (2.2) takes place, we get

$$|a_n| < Kr^{q-n}, \quad r > r_0, \quad q > 0 .$$

Let’s approach r to ∞ . When $n > q$, we get $a_n = 0$. Hence,

$$f(z) = \sum_{n \leq q} a_n z^n .$$

From Liouville's theorem follows, that if an entire function $f(z)$ is not a polynomial, then $M(r)$ when $r \to \infty$ is growing faster than any power r.

The function $M(r)$ takes an important place in the theory of entire functions. Its exact calculating even for the simplest entire functions can cause the difficulties. However, usually, it is enough to be able to evaluate it from above and below.

Let's consider firstly the case of the polynomial of the power $n\,(n \geq 1)$

$$P(z)=a_0+a_1z+\ldots+a_nz^n, \quad a_n\neq 0.$$

In the circle $|z|\leq r$ we have

$$|P(z)|=\left|a_0+a_1z+\ldots+a_nz^n\right|\leq|a_0|+|a_1|\cdot|z|+\ldots+|a_n|\cdot\left|z^n\right|\leq$$

$$\leq|a_0|+|a_1|r+\ldots+|a_n|r^n=|a_n|r^n\left(1+\left[\frac{|a_{n-1}|}{|a_n|}\cdot\frac{1}{r}+\cdots+\frac{|a_0|}{|a_n|}\cdot\frac{1}{r^n}\right]\right).$$

When $r\to\infty$, the sum in the square brackets tends to 0. Therefore, for each $\varepsilon>0$, you can specify $r_0(\varepsilon)$, that when $r>r_0(\varepsilon)$, the inequality

$$|P(z)|\leq|a_n|r^n(1+\varepsilon), \quad |z|\leq r \tag{2.3}$$

is hold.

Let`s consider the value $|P(z)|$ at any point z_0, that located on the circle $|z|=r$. For it $|z_0|=r$. At this point, the inequality (2.3) also takes place, when $r>r_0(\varepsilon)$. On the other hand,

$$|P(z)|=\left|a_0+a_1z+\ldots+a_nz^n\right|\leq|a_0|+|a_1|\cdot|z|+\ldots+|a_n|\cdot\left|z^n\right|\leq$$

$$|P(z_0)|=\left|a_nz_0^n+\left(a_{n-1}z_0^{n-1}+\cdots+a_0\right)\right|\geq\left|a_nz_0^n\right|-\left|a_{n-1}z_0^{n-1}+\cdots+a_0\right|\geq$$

$$\geq|a_n|\cdot|z_0|^n-|a_{n-1}|\cdot|z_0|^{n-1}-\cdots-|a_0|=|a_n|\cdot r^n-|a_{n-1}|\cdot r^{n-1}-\cdots-|a_0|=$$

$$=|a_n|r^n\left(1-\left[\frac{|a_{n-1}|}{|a_n|}\cdot\frac{1}{r}+\cdots+\frac{|a_0|}{|a_n|}\cdot\frac{1}{r^n}\right]\right).$$

Therefore, when $r>r_0(\varepsilon)$, we obtain

$$|P(z_0)|\geq|a_n|r^n(1-\varepsilon), \quad |z_0|=r. \tag{2.4}$$

Returning to (2.3), we conclude that this inequality is also true at the point of the circle $|z|\leq r$, where $|P(z)|$ reaches its maximum $M(r)$. Therefore,

$$M(r)\leq|a_n|r^n(1+\varepsilon), \quad r>r_0(\varepsilon). \tag{2.5}$$

On the other hand, the value $|P(z)|$ at the point z_0 located on the circle $|z|=r$ does not exceed $M(r)$. Hence,

$$M(r) \geq |a_n| r^n (1-\varepsilon), \quad r > r_0(\varepsilon). \tag{2.6}$$

From (2.5) and (2.6) it follows that

$$1-\varepsilon \leq \frac{M(r)}{|a_n| r^n} \leq 1+\varepsilon, \quad r > r_0(\varepsilon),$$

and since ε is an arbitrary positive number, then

$$\lim_{n\to\infty} \frac{M(r)}{|a_n| r^n} = 1.$$

The resulting relation we'll formulate as follows: the maximum modulus of the polynomial of the power n is asymptotically equal to the modulus of the leading term of the polynomial.

Now let's calculate the functions $M(r)$ for e^z, $\cos z$ and $\sin z$. To distinguish them, let's denote $M\left(r, e^z\right)$, $M(r, \cos z)$, $M(r, \sin z)$.

In each of these cases, we use the estimate of the modulus of the sum of the power series

$$\left|e^z\right| = \left|1+\frac{z}{1!}+\frac{z^2}{2!}+\cdots+\frac{z^n}{n!}+\cdots\right| \leq 1+\frac{|z|}{1!}+\frac{|z|^2}{2!}+\cdots+\frac{|z|^n}{n!}+\cdots \leq$$

$$\leq 1+\frac{r}{1!}+\frac{r^2}{2!}+\cdots+\frac{r^n}{n!}+\cdots, \quad |z| \leq r.$$

So, $\left|e^z\right| \leq e^r$ in the disc $|z| \leq r$. But at the point $z = r$, $e^z = e^r$ and $\left|e^z\right| = e^r$. Therefore, at this point, the modulus of the exponential function reaches the maximum value. And

$$M\left(r, e^z\right) = \max_{|z|\leq r} \left\{\left|e^z\right|\right\} = e^r. \tag{2.7}$$

Similarly

$$|\cos z| = \left|1-\frac{z}{1!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots\right| \leq 1+\frac{|z|^2}{2!}+\frac{|z|^4}{4!}+\frac{|z|^6}{6!}+\cdots \leq$$

$$\leq 1+\frac{r^2}{2!}+\frac{r^4}{4!}+\frac{r^6}{6!}+\cdots, \quad |z| \leq r.$$

The sum of the series on the right side of the last inequality is equal to $ch\,r=\frac{1}{2}\left(e^{r}+e^{-r}\right)$. Therefore,

$$\left|\cos z\right|\le\frac{1}{2}\left(e^{r}+e^{-r}\right),\quad |z|\le r.$$

But at the point $z=ir$

$$\cos z=\cos(ir)=\frac{1}{2}\left(e^{i(ir)}+e^{-i(ir)}\right)=\frac{1}{2}\left(e^{-r}+e^{r}\right),\quad |z|\le r,$$

i. e.

$$M(r,\cos z)=\max_{|z|\le r}\{|\cos z|\}=\frac{1}{2}\left(e^{-r}+e^{r}\right). \tag{2.8}$$

Finally, $\left|\sin z\right|\le|z|+\frac{|z|^3}{3!}+\frac{|z|^5}{5!}+\cdots\le r+\frac{r^3}{3!}+\frac{r^5}{5!}+\cdots=sh\,r=\frac{1}{2}\left(e^{r}-e^{-r}\right)$. But at the point $z=ir$, $\left|\sin z\right|=\left|\sin(ir)\right|=\left|\frac{1}{2i}\left(e^{i(ir)}-e^{-i(ir)}\right)\right|=\left|\frac{1}{2i}\left(e^{-r}-e^{r}\right)\right|=\frac{1}{2}\left(e^{r}-e^{-r}\right)$.

Therefore,

$$M(r,\sin z)=\max_{|z|\le r}\{|\sin z|\}=\frac{1}{2}\left(e^{r}-e^{-r}\right). \tag{2.9}$$

Now let's make sure that the functions $M\left(r,e^{z}\right), M(r,\cos z), M(r,\sin z)$ are increasing. For the first function it is obvious. For $M(r,\sin z)$, it follows from the fact that e^{r} is increasing function, and e^{-r} is decreasing function, so the difference $e^{r}-e^{-r}$ is increasing. To check if $M(r,\cos z)$ increases, when $r>0$, it is enough to find its derivative

$$\left(\frac{1}{2}\left(e^{r}+e^{-r}\right)\right)'=\frac{1}{2}\left(e^{r}-e^{-r}\right)>0,\quad r>0.$$

Therefore, $M(r,\cos z)$ is increasing, when $r>0$.

Let $0 \le r < \infty$. Then $\varphi(n) = |a_n| r^n \to 0$, when $n \to \infty$ and there exists $\max\{\varphi(n) : n \ge 0\}$, which we'll denote as $\mu(r)$, such that $\mu(r) = \max\{\varphi(n) : n \ge 0\}$. This value $\mu(r)$ is called the maximum term of the series (2.1). From Cauchy's inequality follows that $\mu(r) \le M(r)$. The last inequality is also called Cauchy's inequality. In the other hand, for each $\alpha \in [0;1]$, we have

$$M(r) \le \sum_{n=0}^{\infty} |a_n| r^n = \sum_{n=0}^{\infty} |a_n| \left(\frac{r}{\alpha}\right)^n \alpha^n \le \mu\left(\frac{r}{\alpha}\right) \sum_{n=0}^{\infty} \alpha^n .$$

So

$$M(r) \le \mu\left(\frac{r}{\alpha}\right) \frac{1}{1-\alpha} . \qquad (2.10)$$

Note that $\mu(r) \to \infty$ when $r \to \infty$.

Order and type of entire function

The value

$$\rho = \overline{\lim_{r\to\infty}} \frac{\ln \ln M(r)}{\ln r} \qquad (2.11)$$

is called the order of the function, and if $0 < \rho < \infty$, the value of

$$\sigma = \overline{\lim_{r\to\infty}} \frac{\ln M(r)}{r^{\rho}} \qquad (2.12)$$

it is called the type of entire function f. It is said that the function f of the order $\rho \left(0 < \rho < \infty\right)$ has minimal, normal, maximum type, if $\sigma = 0, 0 < \sigma < \infty, \sigma = \infty$ correspondingly.

The formulas (2.11), (2.12) allow us using the formulas (2.7), (2.8) and (2.9) to find the order and type of the functions $e^z, \cos z, \sin z$. Since

$$M\left(r, e^z\right) = e^r ,$$

then the order of the function $\rho = 1$ and the type $\sigma = 1$. Next

$$M(r,\cos z)=\frac{1}{2}\left(e^{r}+e^{-r}\right)=e^{r}\cdot\frac{1+e^{-2r}}{2}.$$

Obviously, that $e^{-2r}\to 0\,(r\to\infty)$ and therefore, the fraction on the right side of the last inequality approaching to $\frac{1}{2}$. Let's calculate

$$\ln M(r,\cos z)=r+\ln\frac{1+e^{-2r}}{2}=r\left(1+\frac{1}{r}\ln\frac{1+e^{-2r}}{2}\right),$$

where the expression in parentheses tends to 1, when $r\to\infty$. Once agaon logarithmization will give

$$\ln\ln M(r,\cos z)=\ln r+\ln\left(1+\frac{1}{r}\ln\frac{1+e^{-2r}}{2}\right),$$

where the second term on the right side tends to 0, when $r\to\infty$. Therefore

$$\overline{\lim_{r\to\infty}}\frac{\ln\ln M(r,\cos z)}{\ln r}=\overline{\lim_{r\to\infty}}\left[1+\frac{1}{\ln r}\ln\left(1+\frac{1}{r}\ln\frac{1+e^{-2r}}{2}\right)\right]=1.$$

This means that for the function $\cos z$ we have $\rho=1$ and $\sigma=1$.

We propose to determine independently the order and type of the functions:

a) $f(z)=\sin z$. Answer: $\rho=1,\sigma=1$;

b) $f(z)=e^{z^k}$. Answer: $M\left(r,e^{z^k}\right)=e^{r^k}$, $\rho=k,\sigma=1$;

c) $f(z)=e^{e^k}$. Answer: $M\left(r,e^{e^k}\right)=e^{e^r}$, $\rho=\infty,\sigma$ – uncertain.

In all examples the order of the entire function was equal to the integer number or ∞. But there exists the functions of the fractional orders. For example, $f(z)=\frac{1}{2}\left(e^{\sqrt{z}}+e^{-\sqrt{z}}\right)$. This function is entire, because

$$e^{\sqrt{z}}=1+\frac{\sqrt{z}}{1!}+\frac{z}{2!}+\frac{z\sqrt{z}}{3!}+\cdots,$$

$$e^{-\sqrt{z}}=1-\frac{\sqrt{z}}{1!}+\frac{z}{2!}-\frac{z\sqrt{z}}{3!}+\cdots.$$

And so it is represented everywhere by the convergent power series

$$\frac{1}{2}\left(e^{\sqrt{z}}+e^{-\sqrt{z}}\right)=1+\frac{z}{2!}+\frac{z^2}{4!}+\frac{z^3}{6!}\cdots.$$

$M\left(r,\frac{1}{2}\left(e^{\sqrt{z}}+e^{-\sqrt{z}}\right)\right)=\frac{1}{2}\left(e^{\sqrt{r}}+e^{-\sqrt{r}}\right)$ and it is easy to check, that order $\rho=\frac{1}{2}$ and type $\sigma=1$.

Let's show now that in the defined order and type of the entire function we can take $\mu(r)$ instead of $M(r)$.

Theorem 2.3. *The next equalities take place*

$$\overline{\lim_{r\to\infty}}\frac{\ln\ln\mu(r)}{\ln r}=\rho,\qquad \overline{\lim_{r\to\infty}}\frac{\ln\mu(r)}{r^{\rho}}=\sigma. \tag{2.13}$$

Proof. Let's denote

$$\rho_1=\overline{\lim_{r\to\infty}}\frac{\ln\ln\mu(r)}{\ln r},\quad \sigma_1=\overline{\lim_{r\to\infty}}\frac{\ln\mu(r)}{r^{\rho}}.$$

From the Cauchy's inequality $\mu(r)\le M(r)$ follows that $\rho_1\le\rho$, $\sigma_1\le\sigma$. Next, if we assume that $\rho_1<\infty$, then by definition of ρ_1 we'll get that $\ln\ln\mu(r)\le(\rho_1+\varepsilon)\ln r$ for each $\varepsilon>0$, when $r\ge r_0(\varepsilon)$, that is $\ln\mu(r)\le r^{\rho_1+\varepsilon}$. Therefore, from (2.10) we get

$$\ln M(r)\le\ln\mu\left(\frac{r}{\alpha}\right)+\ln\frac{1}{1-\alpha}\le r^{\rho_1+\varepsilon}\alpha^{-(\rho_1+\varepsilon)}\left(1+o(1)\right),\quad r\to\infty,$$

whence $\ln\ln M(r)\le(\rho_1+\varepsilon)\ln r+O(1)$, when $r\to\infty$. From the last inequality it follows that $\rho\le\rho_1+\varepsilon$, and due to arbitrariness of ε, we have $\rho\le\rho_1$. In the case, when $\rho_1=\infty$, the inequality $\rho\le\rho_1$ is obvious. So, $\rho=\rho_1$.

When $0<\rho<\infty$ and $\sigma_1<\infty$ from the definition of σ_1, we have $\ln\mu(r)\le(\sigma_1+\varepsilon)r^{\rho}$ for each $\varepsilon>0$, when $r\ge r_0(\varepsilon)$ and from (2.10) we obtain $\ln M(r)\le(\sigma_1+\varepsilon)\alpha^{-\rho}r^{\rho}\left(1+o(1)\right)$, $r\to\infty$. Hence, $\sigma\le(\sigma_1+\varepsilon)\alpha^{-\rho}$. Since ε and α

are the arbitrary numbers, then approaching ε to 0 and α to 1, we get $\sigma \le \sigma_1$. When $\sigma_1 = \infty$, the inequality $\sigma \le \sigma_1$is obvious, So, $\sigma = \sigma_1$, and the theorem is proved.

The calculation of the order and type of entire function, using the definition, i. e. from the formulas (2.11) and (2.12), is quite time-consuming, because it is associated with finding the maximum of the modulus of each concrete function. Therefore, there is a natural desire to obtain the formulas for calculating the order and type of the entire function through the coefficients of the series (2.1).

Theorem 2.4. *The next equalities are hold*

$$\rho = \overline{\lim_{n\to\infty}} \frac{n\ln n}{-\ln|a_n|}, \quad \sigma = \overline{\lim_{n\to\infty}} \frac{n}{e\rho}|a_n|^{\rho/n}. \qquad (2\ .14)$$

Proof. Let`s denote

$$\overline{\lim_{n\to\infty}} \frac{n\ln n}{-\ln|a_n|} = k, \quad . \overline{\lim_{n\to\infty}} \frac{n}{e\rho}|a_n|^{\rho/n} = \vartheta$$

From the first equality (2.13) we have that $\mu(r) \le \exp\{r^{\rho+\varepsilon}\}$, when $\rho < \infty$ for each $\varepsilon > 0$, when $r \ge r_0(\varepsilon)$. Then by definition of the $\mu(r)$, we get

$$|a_n| \le \mu(r) r^{-n} \le r^{-n} \exp\{r^{\rho+\varepsilon}\}$$

for all $n \ge 0$ and all $r \ge r_0(\varepsilon)$. Let’s choose $r = n^{1/(\rho+\varepsilon)}$. Then we have that $|a_n| \le \left(e / n^{1/(\rho+\varepsilon)}\right)^n$, when $n \ge n_0$. It follows that $k \le \rho$. When $\rho = \infty$ the last inequality is trivial.

Suppose now that $\sigma < \infty$. From the second equality (2.12) we have $\mu(r) \le \exp\{(\sigma+\varepsilon) r^{\rho}\}$ for each $\varepsilon > 0$, when $r \ge r_0(\varepsilon)$, as before,

$$|a_n| \le r^{-n} \exp\{(\sigma+\varepsilon) r^{\rho}\}$$

for all $n \ge 0$ and $r \ge r_0(\varepsilon)$. Let's choose $r = \left(\dfrac{n}{\rho(\sigma+\varepsilon)}\right)^{1/\rho}$. Then $|a_n| \le \left(e\rho(\sigma+\varepsilon)/n\right)^{n/\rho}$, when $n \ge n_0(\varepsilon)$. Whence it easily follows that $\vartheta \le \sigma$. When $\sigma = \infty$, the inequality $\vartheta \le \sigma$ is trivial.

Let's now prove the opposite inequalities $k \ge \rho$ and $\vartheta \ge \sigma$. Suppose first that $k < \infty$. Then from the definition of k we have $|a_n| \le n^{-n/(k+\varepsilon)}$, when $n \ge n_0(\varepsilon) \ge 1$. Therefore

$$\mu(r) = \max\left\{|a_n| r^n : n \ge 0\right\} \le \max\left\{\max\left\{|a_n| r^n : 0 \le n \le n_0(\varepsilon)\right\}\right\},$$

$$\max\left\{n^{-n/(k+\varepsilon)} r^n : n \ge n_0(\varepsilon)\right\} \le \max\left\{O\left(r^{n_0(\varepsilon)}\right)\right\},$$

$$\exp\left(\max\left\{\varphi(t) : t \ge 1\right\}\right), \quad r \to \infty,$$

where $\varphi(t) = -\dfrac{t \ln t}{k+\varepsilon} + t \ln r$. Since $\varphi'(t) = -\dfrac{\ln t + 1}{k+\varepsilon} + \ln r$ and $\varphi''(t) = -1/\left(t(k+\varepsilon)\right) < 0$, then the maximum of the function φ is reaches at the point $t = \dfrac{1}{e} r^{k+\varepsilon}$, i. e.

$$\ln \mu(r) \le \max\left\{O(\ln r), \frac{r^{k+\varepsilon}}{e(k+\varepsilon)}\right\} = O\left(r^{k+\varepsilon}\right), \quad r \to \infty.$$

According to the first equality (2.13), we get that $\rho \le k$. When $k = \infty$ the inequality $\rho \le k$ is trivial. So, $\rho = k$, and first equality (2.14) is proved.

Suppose that $\vartheta < \infty$. Then $|a_n| \le \left(\dfrac{1}{n} e\rho(\vartheta+\varepsilon)\right)^{n/\rho}$ for each $\varepsilon > 0$, when $n \ge n_0(\varepsilon) \ge 1$ and

$$\mu(r) \le \max\left\{O\left(r^{n_0(\varepsilon)}\right), \exp\left(\max\left\{\varphi(t) : t \ge 1\right\}\right)\right\},$$

where now $\varphi(t) = \dfrac{t}{\rho} \ln\left(\dfrac{1}{t} e\rho(\vartheta+\varepsilon)\right) + t \ln r$.

Since $\varphi'(t)=\ln r-\frac{1}{\rho}+\frac{1}{\rho}\ln\left(\frac{1}{t}e\rho(\vartheta+\varepsilon)\right)$ and $\varphi''(t)=-\frac{1}{\rho t}<0$, the maximum of the modulus of the function φ is reached at the point $t=\rho(\vartheta+\varepsilon)r^{\rho}$ and $\ln\mu(r)\le\max\left\{O(\ln r),(\vartheta+\varepsilon)r^{\rho}\right\}=(1+o(1))(\vartheta+\varepsilon)r^{\rho}, r\to\infty$. Hence, from (2.13) it follows, that $\sigma\le\vartheta$. When $\vartheta=\infty$ the inequality $\sigma\le\vartheta$ is trivial. So, $\sigma=\vartheta$, and second equality (2.14) is proved.

Now consider the question about the derivative and zeros of an entire function. Let's formulate the following obvious statement: 1) if $f(z)$ is the entire function of order $\rho, 0\le\rho<\infty$, then the function $P(z)f(z)$, where $P(z)$ is the polynomial, has the order ρ; 2) if $f(z)$ is the entire function of the order $\rho, 0<\rho<\infty$, and type $\sigma, 0\le\sigma\le\infty$, then the function $P(z)f(z)$ has the type σ and order ρ.

Theorem 2.5. *The function $f(z)$ and its derived $f'(z)$ have the same orders and types.*

Proof. The functions $f'(z)$ *and* $zf'(z)$ have the equal orders and types. Therefore, it is enough to show that functions $f(z)$ and $zf'(z)$ have same orders and types.

Let $f(z)$, represented by the series (1.1), has the order ρ. Then

$$zf'(z)=\sum_{n=1}^{\infty}b_n z^n, \quad b_n=na_n.$$

According to the first of the formulas (2.14), the order of this function is equal to

$$\rho_1=\overline{\lim_{n\to\infty}}\frac{n\ln n}{\ln|1/b_n|}=\overline{\lim_{n\to\infty}}\frac{\ln n}{\ln\sqrt[n]{|1/b_n|}}=\overline{\lim_{n\to\infty}}\frac{\ln n}{\ln\sqrt[n]{1/(na_n)}}=$$

$$=\overline{\lim_{n\to\infty}}\frac{\ln n}{\ln\sqrt[n]{|1/a_n|}-\ln\sqrt[n]{n}}=\overline{\lim_{n\to\infty}}\frac{\ln n}{\ln\sqrt[n]{|1/a_n|}}=\rho.$$

Let $f(z)$ now has the order ρ and type σ. The order of the function $zf'(z)$ is equal to ρ, and its type σ_1 is calculated by the second of the formulas (2.14)

$$\sigma_1 = \overline{\lim_{n\to\infty}} \frac{n}{e\rho}|b_n|^{\rho/n} = \overline{\lim_{n\to\infty}} \frac{n}{e\rho}|na_n|^{\rho/n} = \overline{\lim_{n\to\infty}} \frac{n}{e\rho}|a_n|^{\rho/n}\left(n^{\frac{1}{n}}\right)^{\rho} = \overline{\lim_{n\to\infty}} \frac{n}{e\rho}|a_n|^{\rho/n} = \sigma .$$

The theorem is proved.

Let's remind that a point is called a zero of the function f, if $f(a)=0$. If $f(a)=f'(a)=\cdots=f^{n-1}(a)=0$, and $f^{(n)}(a)\neq 0$, then a is called a zero of the n-th order of the function f. For example, consider the function $\sin z = z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots$. From the series expansion we can see that $z=0$ is a zero of the 1-st order. Now let`s consider the function $z-\sin z=\frac{z^3}{3!}-\frac{z^5}{5!}+\cdots$ $z=0$, for which is obvious that $z=0$ is a zero of the third order.

Let $\lambda_1,\lambda_2,\ldots,\lambda_n,\ldots$ is the sequence of the complex numbers, and $|\lambda_n|\uparrow\infty,\ \lambda_1\neq 0$. Suppose that there exists a finite α, such that

$$\sum_{n=1}^{\infty}\frac{1}{|\lambda_n|^{\alpha}}<\infty \quad . \tag{2.15}$$

The lower bound τ of the set of numbers α, that satisfy the condition (2.15), is called the convergence indicator of the sequence $\{\lambda_n\}$. If τ is finite, then for each $\varepsilon>0$

$$\sum_{n=1}^{\infty}\frac{1}{|\lambda_n|^{\tau+\varepsilon}}<\infty, \qquad \sum_{n=1}^{\infty}\frac{1}{|\lambda_n|^{\tau-\varepsilon}}=\infty .$$

If α does not exist, when the condition (2.15) is hold, then $\tau=\infty$.

Let $f(z)$ be the entire function of the finite order ρ and $\lambda_1,\lambda_2,\ldots,\lambda_n,\ldots$ are its zeros, that differs from the beginning of the coordinates. We suppose, that $|\lambda_n|\uparrow\infty$ and each zero is written as many times in the sequence $\{\lambda_n\}$, which is its multiplicity.

Theorem 2.6. *Let $f(z)$ be the entire function of the finite order ρ and $\lambda_1,\lambda_2,\ldots,\lambda_n,\ldots$ $|\lambda_n|\uparrow\infty,\ \lambda_1\neq 0$ are its zeros. Then $\tau\leq\rho$.*

Proof. We suppose that $f(0)\neq 0$ (otherwise we would divide $f(z)$ by the corresponding power of z and get the function of order ρ, which doen't equal zero at the beginning of the coordinates and for which $\lambda_1,\lambda_2,\ldots,\lambda_n$ are zeros). Denote $r_n=|\lambda_n|,\ R=2r_n$. The function $f(z)$ has at least n zeros $\lambda_1,\lambda_2,\ldots,\lambda_n$, in the circle $|z|\leq R$. Then, by the well-known estimate

$$2^n=\left(\frac{R}{r_n}\right)^n\leq\frac{R^n}{\left|\lambda_1^0\ldots\lambda_n\right|}\leq\frac{M(2r_n)}{|f(0)|}<e^{r_n^{\rho+\varepsilon}},\quad n>n_0,$$

when n is large enough, then we have

$$n\ln 2<r_n^{\rho+\varepsilon},\ n<r_n^{\rho+2\varepsilon},\ r_n>n^{1/(\rho+2\varepsilon)},\ r_n^{\rho+3\varepsilon}>n^{\beta},\ \beta=\frac{\rho+3\varepsilon}{\rho+2\varepsilon}>1.$$

So, $\sum\limits_{n=1}^{\infty}\frac{1}{|\lambda_n|^{\rho+3\varepsilon}}<\infty$, where $\varepsilon>0$ is arbitrary number. Therefore, $\tau\leq\rho$.

Infinite products

The expression

$$(1+a_1)(1+a_2)\ldots(1+a_n)\ldots,\quad a_n\neq -1\quad (n\geq 1) \tag{2.16}$$

is called the infinite product and is denoted $\prod\limits_{n=1}^{\infty}(1+a_n)$. The expression $p_n=\prod\limits_{k=1}^{n}(1+a_k)$ is called the partial product. It is said that the infinite product converges, if there exists $\lim\limits_{n\to\infty}p_n=p\neq 0,\infty$. Otherwise, it is called the divergent product. If it is convergent, then denote $p=\prod\limits_{n=1}^{\infty}(1+a_n)$.

Necessary condition of convergence. *If the product is convergent, then* $a_n\to 0, n\to\infty$.

Indeed, $1+a_n=\frac{p_n}{p_{n-1}}\to 1\left(n\to\infty\right)$ and it follows that $a_n\to 0\left(n\to\infty\right)$.

Sufficient condition of convergence. *If* $a_n\geq 0\left(n\geq 1\right)$*, then the product (2.16) and the series* $\sum_{n=1}^{\infty}a_n$ *are convergent or divergent simultaneously.*

Indeed, since in this case p_n is non-decreasing function of n, then p_n is tending either to the finite limit or to ∞. Further, from $1+a_n\leq e^{a_n}$ we have $a_1+a_2+\ldots+a_n\leq(1+a_1)(1+a_2)\ldots(1+a_n)\leq e^{a_1+a_2+\ldots+a_n}$. These inequalities show that p_n and $a_1+a_2+\ldots+a_n$ are bounded or unbounded at the same time, and this completes the proof.

The product (2.16) is absolutely convergent, if the product

$$\prod_{n=1}^{\infty}\left(1+|a_n|\right) \qquad (2.17)$$

is convergent. The product is absolutely convergent if and only if the series $\sum_{n=1}^{\infty}|a_n|$ is convergent. If the product (2.17) is divergent, then (2.16) is conditionally convergent.

Let's show that the absolutely convergent product is convergent. Let's denote

$$p_n=\prod_{k=1}^{n}(1+a_k),\quad P_n=\prod_{k=1}^{n}\left(1+|a_k|\right).$$

We have

$$p_n-p_{n-1}=(1+a_1)(1+a_2)\ldots(1+a_{n-1})a_n,\quad P_n-P_{n-1}=\left(1+|a_1|\right)\ldots\left(1+|a_{n-1}|\right)|a_n|,$$

so, $\left|p_n-p_{n-1}\right|\leq P_n-P_{n-1}$. Let the product is absolutely convergent. Then there exists $\lim_{n\to\infty}P_n=P\neq\infty$ and, hence, the series $\sum_{n=2}^{\infty}\left(P_n-P_{n-1}\right)$ is convergent. But then the series $\sum_{n=2}^{\infty}\left(p_n-p_{n-1}\right)$ is convergent too. Therefore, there exists $\lim_{n\to\infty}p_n=p\neq\infty$.

Let's make sure that $p \neq 0$. Indeed, the series $\sum_{n=1}^{\infty} |a_n|$ is convergent and $1 + a_n \to 1, n \to \infty$. Therefore, the series $\sum_{n=1}^{\infty} \left(1 - a_n / (1 + a_n)\right)$ converges. Hence, the product

$$\prod_{k=1}^{n} \left(1 - \frac{a_k}{1 + a_k}\right)$$

tends to some finite limit. But this product is equal to $\frac{1}{p_n}$. So, the limit of the product p_n is different from zero.

Note that the convergence of the product represented in the form $\prod_{n=1}^{\infty} u_n$ is equivalent to the convergence of the series $\sum_{n=n_0}^{\infty} \ln u_n$.

Now let's consider the functional infinite products

$$\prod_{n=1}^{\infty} \left(1 + u_n(z)\right), \tag{2.18}$$

where $u_n(z)$ are the functions defined in the domain D, and $u_n(z) \neq -1$ in D.

The product (2.18) is uniformly convergent in the domain D, if it is convergent at every point $z \in D$. We have

$$\sum_{n=1}^{\infty} |u_n(z)| < \infty, \quad z \in D,$$

then the product (2.18) is convergent in the domain D. The product (2.18) is convergent to $p(z)$ at every point $z \in D$ and we write

$$p(z) = \prod_{n=1}^{\infty} \left(1 + u_n(z)\right).$$

The product (2.18) is called uniformly convergent in the domain D, if the sequence of partial products $p_n(z)=\prod_{k=1}^{n}(1+u_k(z))$ converges in the domain D. The infinite product is called uniformly convergent inside of the domain D, if the sequence of partial products converges uniformly inside the domain D.

A sufficient sign of the uniform convergence in the domain. *If in the domain D*

$$|u_n(z)| \le a_n \quad (n \ge 1), \quad \sum_{n=1}^{\infty} a_n < \infty ,$$

then in the domain D the product (2.18) converges uniformly.

Indeed, $p_n(z)-p_{n-1}(z) \le (1+u_1(z))\dots(1+u_{n-1}(z))u_n(z)$

$$|p_n(z)-p_{n-1}(z)| \le (1+|u_1(z)|)\dots(1+|u_{n-1}|(z))|u_n(z)| \le$$

$$\le e^{a_1+\dots+a_n} \cdot a_n < K|a_n|, \quad K = const ,$$

therefore, the series $\sum_{n=2}^{\infty}(p_n(z)-p_{n-1}(z))$ converges uniformly in D, hence, $\{p_n(z)\}$ converges uniformly in D.

A sufficient sign of the uniform convergence in the domain. *Let $u_n(z)$ are the analytical functions in a simple connected domain D. If the series*

$$\sum_{n=1}^{\infty} \ln(1+u_n(z)) \tag{2.19}$$

converges uniformly in D, then the product (2.18) converges uniformly inside the domain D.

Indeed, we have

$$\prod_{k=1}^{n}(1+u_k(z)) = \exp\left[\sum_{k=1}^{n}\ln(1+u_k(z))\right], \quad z \in D .$$

And we get

$$p(z)=\prod_{k=1}^{\infty}(1+u_k(z)) = \exp\left[\sum_{k=1}^{\infty}\ln(1+u_k(z))\right], \quad z \in D .$$

The right part is the analytical function in D, and therefore $p(z)$ is the analytical function in D. Next

$$p(z)-\prod_{k=1}^{n}\left(1+u_k(z)\right)=\exp\left[\sum_{k=1}^{\infty}\ln\left(1+u_k(z)\right)\right]-\exp\left[\sum_{k=1}^{n}\ln\left(1+u_k(z)\right)\right]=$$

$$=p(z)\left\{1-\exp\left[-\sum_{k=n+1}^{\infty}\ln\left(1+u_k(z)\right)\right]\right\}.$$

Let K is some compact in D. The function $p(z)$ is bounded on the compact, i. e. $|p(z)|\le M,\, z\in K$. The series (1.19) converges uniformly on the compact K, so

$$\left|\sum_{k=n+1}^{\infty}\ln\left(1+u_k(z)\right)\right|<\frac{\varepsilon}{2M}<\frac{1}{2},\quad z\in K,\quad n>N.$$

Hence,

$$\left|p(z)-\prod_{k=1}^{n}\left(1+u_k(z)\right)\right|\le M\left\{\frac{\varepsilon}{2M}+\frac{1}{2!}\left(\frac{\varepsilon}{2M}\right)^2+\cdots\right\}\le$$

$$\le\frac{\varepsilon}{2}\left\{1+\frac{1}{2!}\cdot\frac{\varepsilon}{2M}+\frac{1}{3!}\cdot\left(\frac{\varepsilon}{2M}\right)^2+\cdots\right\}<\frac{\varepsilon}{2}\left\{1+\frac{1}{2}+\frac{1}{2^2}+\cdots\right\}=\varepsilon,\quad z\in K, n>N,$$

that proves the uniform convergence (2.18) inside D.

Until now, we assumed that $u_n(z)\ne -1$ in D. Let's assume that D is a simple connected domain, $u_n(z)$ are the analytical functions in D, which have the following property: whatever $K\subset D$, there is a number $N_0=N_0(K)$, such that $u_n(z)\ne -1,\, z\in K, n>N_0$.

We assume that the product (2.18) converges in D, if the product $\prod_{n=N}^{\infty}\left(1+u_n(z)\right),\, z\in K$ converges, whatever the compact $K\subset D$. Then the function $p(z)=\prod_{n=1}^{\infty}\left(1+u_n(z)\right)$ can convert to zero at some points of the domain D, at the point of such type at least one of the multipliers of the product will convert to zero.

Then a sufficient sign of the uniform convergence in the domain D will be formulated as follows: if the series $\sum_{n=N_0}^{\infty} \ln\left(1+u_n(z)\right),\, N_0 = N_0(K)$ converges uniformly on the arbitrary compact $K \subset D$, then the product will converge uniformly inside the domain D.

Decomposition of entire function into infinite product

Let's firstly consider the method of constructing the entire function with the given zeros. Let's denote

$$E(u,0)=1-u, \quad E(u,n)=(1-u)\exp\left\{u+\frac{u^2}{2}+\cdots\frac{u^n}{n}\right\}, \quad n>1.$$

These expressions are called the primary multipliers. Their behavior depends upon n, when $u \to \infty$.

Lema 2.1. *The next inequalities are hold*

$$\left|\ln\left|E(u,n)\right|\right| \le \left|\ln E(u,n)\right| \le 2|u|^{n+1}, \quad |u| \le \frac{1}{2}, \tag{2.20}$$

$$\left|\ln\left|\exp\left\{u+\frac{u^2}{2}+\cdots\frac{u^n}{n}\right\}\right|\right| \le \left(2|u|\right)^n, \quad |u| \ge \frac{1}{2}. \tag{2.21}$$

Proof. We have

$$\ln E(u,n) = \ln(1-u) + u + \frac{u^2}{2} + \cdots + \frac{u^n}{n} = \left(-u - \frac{u^2}{2} - \cdots - \frac{u^n}{n} - \frac{u^{n+1}}{n+1} - \cdots\right) +$$

$$+u + \frac{u^2}{2} + \cdots + \frac{u^n}{n} = -\frac{u^{n+1}}{n+1} - \frac{u^{n+2}}{n+2} - \cdots, \quad |u| \le \frac{1}{2},$$

whence

$$\ln E(u,n) \le |u|^{n+1} + |u|^{n+2} + \cdots = |u|^{n+1}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right) = 2|u|^{n+1}.$$

Now let's prove the second inequality. Let $|u|\ge\frac{1}{2}$. We have

$$e^{-|u|-\frac{|u|^2}{2}-\cdots-\frac{|u|^n}{n}}\le\left|e^{u+\frac{u^2}{2}+\ldots+\frac{u^n}{n}}\right|\le e^{|u|+\frac{|u|^2}{2}+\ldots+\frac{|u|^n}{n}},$$

$$|u|+\frac{|u|^2}{2}+\cdots+\frac{|u|^n}{n}\le\left(|u|^{1-n}+\cdots+1\right)|u|^n\le\left(2^{n-1}+\cdots+1\right)|u|^n\le 2^n|u|^n.$$

Therefore, $e^{-(2|u|)^n}\le\left|e^{u+\frac{u^2}{2}+\ldots+\frac{u^n}{n}}\right|\le e^{(2|u|)^n}$.

Whence

$$\left|\ln\left|e^{u+\frac{u^2}{2}+\cdots\frac{u^n}{n}}\right|\right|\le\left(2|u|\right)^n,\quad |u|\ge\frac{1}{2}.$$

Theorem 2.7. *Let* $|z_1|\le|z_2|\le\cdots$, $\lim\limits_{n\to\infty}z_n=\infty$, *and let the entire* p_n *are such that for the arbitrary* r

$$\sum_{n=1}^{\infty}\left(\frac{r}{r_n}\right)^{p_n}<\infty,\quad r_n=|z_n|.$$

Then the product

$$\prod_{n=1}^{\infty}E\left(\frac{z}{z_n},p_n-1\right)=\prod_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right)\exp\left\{\frac{z}{z_n}+\frac{z^2}{2z_n^2}+\cdots+\frac{z^{p_n-1}}{(p_n-1)z_n^{p_n-1}}\right\}\qquad(2.22)$$

converges in all plane and defines the entire function $F(z)$, *which has zeros at the points* $z_1,z_2,\ldots$ *and only in them.*

Proof. Let $R>0$, $|z|\le R$ and $|z_n|\le 2R$, when $n>N$. From the inequality (2.20) follows that

$$\left|\ln E\left(\frac{z}{z_n},p_n-1\right)\right|\le 2\left(\frac{R}{r_n}\right)^{p_n},\quad |z|\le R, n>N,$$

so, the series $\sum\limits_{n=N}^{\infty} \ln E\left(\frac{z}{z_n}, p_n - 1\right)$ converges uniformly in $|z| \le R$ and then the product $\prod\limits_{n=N}^{\infty} E\left(\frac{z}{z_n}, p_n - 1\right)$ converges uniformly in $|z| \le R$ to the analytical function in $|z| \le R$, that doesn't equal zero. Then the product (2.22) also converges to the analytical function $F(z)$ in $|z| \le R$, moreover this function converts into zero only at the points z_j, that lie in the circle of the radius R. Since R is an arbitrary number, so the theorem is proved.

Notice that we can take $p_n = n\,(n \ge 1)$.

Theorem 2.8. *Let $f(z)$ is the entire function, $z_1, z_2, \ldots$ are zeros (moreover each of them is written out as many times as its multiplicity shows), that differs from the beginning of the coordinates. Let`s choose the integer p_n, such that for the arbitrary r the condition*

$$\sum_{n=1}^{\infty} \left(\frac{r}{r_n}\right)^{p_n} < \infty, \quad r_n = |z_n|$$

is hold. Then

$$f(z) = z^{\lambda} e^{h(z)} \prod_{n=1}^{\infty} E\left(\frac{z}{z_n}; p_n - 1\right),$$

where $h(z)$ is the entire function.

Proof. Let's form a product (2.22) and let it converges to $F(z)$. Denote $H(z) = f(z) / z^{\lambda} F(z)$, where λ is the multiplicity of zero $z = 0$. This is the entire function that is not converted anywhere to zero. Hence, $h(z) = \ln H(z) = \int\limits_0^z \frac{H'(\xi)}{H(\xi)} d\xi + \ln H(0)$ is the entire function. But then

$$f(z) = z^{\lambda} H(z) F(z) = z^{\lambda} e^{h(z)} F(z),$$

and the theorem is proved.

Let $f(z)$ is the entire function of the order ρ and $z_1, z_2, \ldots, z_m \neq 0$ are its zeros. And τ is the convergence indicator of the sequence $\{z_m\}$. According to the Theorem 2.6, we have $\tau \leq \rho$. Let k is the smallest integer, that satisfies the condition

$$\sum_{m=1}^{\infty} \frac{1}{|z_m|^{k+1}} < \infty . \tag{2.23}$$

Then $\sum_{m=1}^{\infty} \left(\frac{r}{r_m} \right)^{k+1} < \infty, \quad r_m = |z_m|$ for arbitrary r. By the Theorem 1.8, we get

$$f(z) = z^{\lambda} e^{h(z)} \prod_{m=1}^{\infty} \left(1 - \frac{z}{z_m} \right) \exp \left\{ \frac{z}{z_m} + \frac{z^2}{2z_m^2} + \cdots + \frac{z^k}{k z_m^k} \right\}, \tag{2.24}$$

where $h(z)$ is the polynomial of the degree $h \leq \rho$. The expression

$$F(z) = \prod_{m=1}^{\infty} \left(1 - \frac{z}{z_m} \right) \exp \left\{ \frac{z}{z_m} + \frac{z^2}{2z_m^2} + \cdots + \frac{z^k}{k z_m^k} \right\} \tag{2.25}$$

is called the canonical product.

Theorem 2.9. *The order of the canonical product (2.25) is equal to* τ *and if*

$$\sum_{m=1}^{\infty} \frac{1}{|z_m|^{\tau}} < \infty , \tag{2.26}$$

then $F(z)$ *is the entire function of the order* τ *of the finite type.*

Proof. In (2.25) the integer k is associated with τ by ratio $k \leq \tau \leq k+1$. Let θ be the number that satisfies the conditions, $\tau \leq \theta \leq k+1$ and $\sum_{m=1}^{\infty} 1/|z_m|^{\theta} < \infty$. This θ can be chosen very close to τ and in the case (2.26) we will accept $\theta = \tau$. By the Lemma 2.1 we have

$$|E(u,k)| \leq e^{(2|u|)^{k+1}} \leq e^{(2|u|)^{\theta}}, \quad |u| \leq \frac{1}{2},$$

$$\left| e^{u + \frac{u^2}{2} + \cdots + \frac{u^k}{k}} \right| \leq e^{(2|u|)^{k}} \leq e^{(2|u|)^{\theta}}, \quad |u| \geq \frac{1}{2}.$$

From the last inequality it follows that

$$\left|E(u,k)\right| \le e^{(2|u|)^{\theta}+\ln(1+|u|)} \le e^{(b|u|)^{\theta}}, \quad b<\infty, \quad |u| \ge \frac{1}{2}.$$

So, for arbitrary u we have $\left|E(u,k)\right| \le e^{(b|u|)^{\theta}}$. Hence, we get

$$\left|F(z)\right| \le \exp\left\{ b\sum_{m=1}^{\infty}\left|\frac{z}{z_m}\right|^{\theta} \right\} = \exp\left\{ a|z|^{\theta} \right\}, \quad a<\infty. \tag{2.27}$$

Therefore, the order of the function $F(z)$ $\rho_F \le \theta$. But θ is very close to τ. Hence, $\rho_F \le \tau$. By the Theorem 2.6 $\tau \le \rho_F$. Then $\tau = \rho_F$.

In the case of (2.26) we will accept that $\theta = \tau$ and then from (2.27) it is clear that $F(z)$ is the entire function of the order τ of the finite type (the fact that it has the order τ we have already proved earlier).

Let's formulate without the proving the basic theorem on the decomposition of the entire function of the finite order into the infinite product.

Theorem 2.10 (Borel's). *Let $f(z)$ be the entire function of the finite order ρ, $z_1, z_2, \ldots$ are its zeros, τ be the indicator of convergence of the sequence $\{z_m\}$, k is the smallest integer, that satisfies the condition*

$$\sum_{m=1}^{\infty}\frac{1}{|z_m|^{k+1}} < \infty.$$

Then the representation (2.24) takes place, moreover $\rho = \max\{h, \tau\}$.

Corollary 2.1. *If ρ is not an integer, then $\tau = \rho$.*

Control questions

1. Formulate the Liouville's theorem.
2. Formulate the definition of the order and the type of the function.
3. Give the formulas for calculation of the order and type of the function using the coefficients of the series.
4. What is the zero of the function?
5. Which function is called an entire?

6. Formulate the definitions of the the maximum of the modulus and the maximum term of the entire function.
7. Formulate a sufficient sign of the convergence of the numerical product.
8. Formulate a sufficient sign of the uniform convergence of the functional infinite product in the domain.
9. Formulate Borrell's theorem of the decomposition of the entire function of the finite order into the infinite product.

Examples of problem solving

1. Find the order and type of the function $\sum_{n=1}^{\infty} e^{-2n\ln n} z^n$.

We have

$$\rho = \overline{\lim_{n\to\infty}} \frac{n\ln n}{-\ln|a_n|} = \overline{\lim_{n\to\infty}} \frac{n\ln n}{z_n \ln n} = \frac{1}{2},$$

$$\sigma = \overline{\lim_{n\to\infty}} \frac{n}{e\rho}|a_n|^{\rho/n} = \overline{\lim_{n\to\infty}} \frac{2n}{e}\left(e^{-2n\ln n}\right)^{\frac{1}{2n}} = \frac{2}{e}.$$

2. Do the decomposition of the function $\sin z$ into the infinite product.

Zeros of this function are $\pm k\pi\ (k=0,1,2,\ldots)$, $\rho=1$, $\tau=1$, $k=1$. The product of the multipliers, that correspond to zeros $k\pi$ and $-k\pi$, is equal to

$$\left(1-\frac{z}{k\pi}\right)e^{\frac{z}{k\pi}}\left(1+\frac{z}{k\pi}\right)e^{-\frac{z}{k\pi}} = 1-\frac{z^2}{k^2\pi^2}.$$

Therefore

$$\sin z = ze^{Az+B}\prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2\pi^2}\right).$$

From the equality $\dfrac{\sin z}{z}=e^{Az+B}\prod_{k=1}^{\infty}\left(1-\dfrac{z^2}{k^2\pi^2}\right)$, when $z\to 0$, we obtain $e^B=1$, i. e. $B=0$. Hence,

$$\sin z = ze^{Az}\prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2\pi^2}\right).$$

The function $\sin z$ is odd, so $e^{Az}=e^{-Az}$ and $e^{2Az}=1$. So, $A=0$ and

$$\sin z = z\prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2\pi^2}\right).$$

3. Investigate the convergence of the product $\prod_{n=1}^{\infty}\left(1+\dfrac{1}{2^n+1}\right)$.

Let's consider the corresponding series $\sum_{n=1}^{\infty}\dfrac{1}{2^n+1}$. Since $\dfrac{1}{2^n+1}<\dfrac{1}{2^n}$, and the series $\sum_{n=1}^{\infty}\dfrac{1}{2^n}$ converges as an infinitely decreasing geometric progression, hence, the product will be convergent.

4. Investigate the convergence of the product $\prod_{n=1}^{\infty}\left(1+\dfrac{z^{\frac{n(n-1)}{2}}}{n!}\right)$. Let's apply the sign of D'Alembert to the corresponding series $\sum_{n=1}^{\infty}\dfrac{z^{\frac{n(n-1)}{2}}}{n!}$:

$$\frac{u_{n+1}}{u_n}=\frac{|z|^n}{n+1};\quad \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\begin{cases}0, & |z|\le 1,\\ \infty, & |z|>1.\end{cases}$$

So, the series and, accordingly, the product converges for $|z|\le 1$.

Tasks for independent work

1. Using the formulas (2.13), determine the orders and types of the functions ($n \in N$):

a) $f(z) = c_0 z^n + c_1 z^{n-1} + \cdots + c_n$. Answer: $\rho = 0$;

b) $f(z) = e^{az^n}\ (a > 0)$. Answer: $\rho = n, \sigma = a$;

c) $f(z) = z^n e^{3z}$. Answer: $\rho = 1, \sigma = 3$;

d) $f(z) = z^2 e^{2z} + e^{3z}$. Answer: $\rho = 1, \sigma = 3$;

e) $f(z) = e^{5z} - 3e^{2z^3}$. Answer: $\rho = 3, \sigma = 2$;

f) $f(z) = e^z \cos z$. Answer: $\rho = 1, \sigma = 2$;

g) $f(z) = \cos\sqrt{z}$. Answer: $\rho = \frac{1}{2}, \sigma = 1$.

2. Using the formulas (2.14), determine the orders and types of the functions:

a) $f(z) = \sum_{n=1}^{\infty} \left(\frac{z}{n}\right)^n$. Answer: $\rho = 1, \sigma = \frac{1}{e}$;

b) $f(z) = \sum_{n=1}^{\infty} \left(\frac{1}{n}\ln n\right)^{n/a} z^n\ (a > 0)$. Answer: $\rho = a, \sigma = \infty$;

c) $f(z) = \sum_{n=2}^{\infty} \left(\frac{1}{n}\ln n\right)^{n/a} z^n\ (a > 0)$. Answer: $\rho = a, \sigma = 0$;

d) $f(z) = \sum_{n=0}^{\infty} e^{-n^2} z^n$. Answer: $\rho = 0$;

e) $f(z) = \sum_{n=1}^{\infty} z^n / n^{n^{1+a}}\ (a > 0)$. Answer: $\rho = 0$;

f) $f(z) = \sum_{n=1}^{\infty} \alpha^{n^2} z^n\ (0 < \alpha < 1)$. Answer: $\rho = 0$;

g) $f(z)=\sum_{n=1}^{\infty} z^n / n^{\alpha n}\,(a>0)$. Answer: $\rho=\dfrac{1}{2}, \quad \sigma=\dfrac{\alpha}{e}$.

3. Investigate the following infinite products for convergence:

a) $\prod_{n=3}^{\infty}\left(n^2-4\right)/\left(n^2-1\right)$. Answer: it converges;

b) $\prod_{n=1}^{\infty}\left(1+1/n(n+2)\right)$. Answer: it converges;

c) $\prod_{n=1}^{\infty}\dfrac{(2n+1)(2n+7)}{(2n+3)(2n+5)}$. Answer: it converges;

d) $\prod_{n=1}^{\infty}\dfrac{1}{n}$. Answer: it diverges;

e) $\prod_{n=1}^{\infty}(n+1)^2/n(n+2)$. Answer: it converges;

f) $\prod_{n=1}^{\infty}\left(1+1/n^p\right)$. Answer: it converges when $p>1$;

g) $\prod_{n=1}^{\infty}\sqrt[n]{1+1/n}$. Answer: it converges;

h) $\prod_{n=1}^{\infty}\sqrt[n^2]{n}$. Answer: it converges.

4. Investigate for absolute convergence:

a) $\prod_{n=1}^{\infty}\left[1+\dfrac{(-1)^{n+1}}{n}\right]$. Answer: it conditionally converges;

b) $\prod_{n=1}^{\infty}\left[1+\dfrac{(-1)^{n+1}}{\sqrt{n}}\right]$. Answer: it conditionally converges;

c) $\prod_{n=1}^{\infty}\left[1+\frac{(-1)^{n+1}}{n^{p}}\right]$. Answer: it converges absolutely, when $p>1$ and conditionally converges, when $\frac{1}{2}<p\le 1$;

d) $\prod_{n=2}^{\infty}\left[1+\frac{(-1)^{n}}{\ln n}\right]$. Answer: it conditionally converges;

e) $\prod_{n=2}^{\infty}\sqrt{n}\,/\left(\sqrt{n}+(-1)^{n}\right)$. Answer: it diverges;

f) $\prod_{n=1}^{\infty}n^{(-1)^{n}}$. Answer: it diverges;

g) $\prod_{n=1}^{\infty}\sqrt[n]{n^{(-1)^{n}}}$. Answer: it conditionally converges;

h) $\prod_{n=1}^{\infty}\left[1+\frac{1}{n}(-1)^{\frac{1}{2}n(n-1)}\right]$. Answer: it conditionally converges.

5. Find the areas of convergence of products:

a) $\prod_{n=1}^{\infty}\left(1-z^{n}\right)$. Answer: $|z|<1$;

b) $\prod_{n=1}^{\infty}\left(1+\frac{z^{n}}{2^{n}}\right)$. Answer: $|z|<2$;

c) $\prod_{n=1}^{\infty}\left(1-\frac{z^{2}}{n^{2}}\right)$. Answer: $|z|<\infty$;

d) $\prod_{n=2}^{\infty}\left(1-\left(1-\frac{1}{n}\right)^{-n}z^{-n}\right)$. Answer: $|z|>1$;

e) $\prod_{n=1}^{\infty}\left(1+\left(1+\frac{1}{n}\right)^{n^{2}}z^{n}\right)$. Answer: $|z|<\frac{1}{e}$;

f) $\prod_{n=1}^{\infty} \cos\frac{z}{n}$. Answer: $|z|<\infty$;

g) $\prod_{n=1}^{\infty} \frac{n}{z}\sin\frac{z}{n}$. Answer: $|z|<\infty$;

h) $\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}$. Answer: $|z|<\infty$.

6. Prove that following decompositions take place:

a) $\cos z = \prod_{k=0}^{\infty}\left(1-4z^2/\pi^2(2k+1)^2\right)$;

b) $shz = z\prod_{k=1}^{\infty}\left(1+z^2/k^2\pi^2\right)$;

c) $chz = \prod_{k=0}^{\infty}\left(1+4z^2/\pi^2(2k+1)^2\right)$ c)

d) $e^z-1=ze^{\frac{z}{2}}\prod_{k=1}^{\infty}\left(1+z^2/4k^2\pi^2\right)$.

Chapter III. Dirichlet Series

Convergence area, convergence abscissas

Let (λ_n) be the increasing to the $+\infty$ sequence of the positive numbers and $\lambda_n = 0$, and (a_n) be the sequence of complex numbers. The series

$$F(s) = a_0 + \sum_{n=1}^{\infty} a_n \exp(s\lambda_n) \quad (s = \sigma + it) \tag{3.1}$$

is called the Dirichlet series (or exponent series), the numbers λ_n are its indicators, and a_n – the coefficients.

If in Taylor's series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z = re^{i\varphi}, \tag{3.2}$$

then we'll obtain the Dirichlet series

$$F(s) = f\left(e^s\right) = \sum_{n=0}^{\infty} a_n \exp(sn) \tag{3.1'}$$

$\lambda_n = n$. Thus, the Dirichlet series is a generalization of the Taylor series.

$$\varsigma(-s) = \sum_{n=1}^{\infty} n^s = \sum_{n=1}^{\infty} \exp(s \ln n) \ . \tag{3.3}$$

The function $\varsigma(s)$ is called the Riemann function and plays an important role in the number theory.

To find the radius R of convergence of the series (3.2), the Cauchy-Hadamar formula is used

$$\frac{1}{R} = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} \quad . \tag{3.4}$$

If $R>0$, then on each compact from the circle $\{z:|z|<R\}$ the series (3.2) is convergent absolutely and uniformly. It follows that the series ($3.1'$) is convergent absolutely and uniformly on each compact from the half-plane $\{s:\operatorname{Re}s<\sigma_ç\}$. From (3.4) it follows that

$$\sigma_ç=-\overline{\lim_{n\to\infty}}\frac{1}{n}\ln|a_n|=\underline{\lim_{n\to\infty}}\frac{1}{n}\ln\frac{1}{|a_n|}.$$

The information about the series ($3.1'$) cannot be automatically transferred to the series (3.1) with the arbitrary indicators. In generally, the series (3.1) may be convergent in some half-plane, but not absolutly convergent.

Let`s A_n and B_n be the arbitrary values, and $C_n=A_p+\ldots+A_n, n\geq p$. Then $A_p=C_p, A_n=C_n-C_{n-1}, n>p$ and

$$\sum_{n=p}^{q}A_nB_n=C_pB_p+B_{p+1}\left(C_{p+1}-C_p\right)+B_{p+2}\left(C_{p+2}-C_{p+1}\right)+\cdots+$$

$$+B_q\left(C_q-C_{q-1}\right)=C_p\left(B_p-B_{p+1}\right)+C_{p+1}\left(B_{p+1}-B_{p+2}\right)+\cdots+C_{q-1}\left(B_{q-1}-B_q\right)+$$

$$+C_qB_q=\sum_{n=p}^{q-1}\left(B_n-B_{n+1}\right)C_n+B_qC_q\ . \tag{3.5}$$

The formula (3.5) is called the Abel transformation and with its help we will prove the following theorem, which is the analogous to the well-known Abel lemma for the power series.

Theorem 3.1. *If the Dirichlet series (3.1) is convergent at the point* $s_0\in C$, *then it is convergent in the half-plane* $\{s:\operatorname{Re}s<\operatorname{Re}s_0\}$ *and uniformly convergent in the closed angle*

$$\left\{s:\left|\arg\left(s-s_0\right)-\pi\right|\leq\gamma\right\},\quad 0<\gamma<\frac{\pi}{2}\ . \tag{3.6}$$

Proof. Using the formula (3.5), we obtain

$$\sum_{n=p}^{q}a_ne^{s\lambda_n}=\sum_{n=p}^{q}a_ne^{s_0\lambda_n}e^{\left(s-s_0\right)\lambda_n}=$$

$$= \sum_{n=p}^{q} \left(e^{(s-s_0)\lambda_n} - e^{(s-s_0)\lambda_{n+1}} \right) \sum_{j=p}^{n} a_j e^{s_0\lambda_j} + e^{(s-s_0)\lambda_q} \sum_{j=p}^{q} a_j e^{s_0\lambda_j} . \tag{3.7}$$

Let $s_0 = \sigma_0 + it_0$. If $\sigma < \sigma_0$, then

$$\left| e^{(s-s_0)\lambda_n} - e^{(s-s_0)\lambda_{n+1}} \right| = \left| (s-s_0) \int_{\lambda_n}^{\lambda_{n+1}} e^{(s-s_0)x} dx \right| \leq$$

$$\leq |s-s_0| \int_{\lambda_n}^{\lambda_{n+1}} e^{(\sigma-\sigma_0)x} dx = \frac{|s-s_0|}{\sigma_0-\sigma} \left[e^{(\sigma-\sigma_0)\lambda_n} - e^{(\sigma-\sigma_0)\lambda_{n+1}} \right].$$

Since the series (3.1) is convergent at the point s_0, then for each $\varepsilon > 0$ there exists $n_0 \in N$, such that

$$\left| \sum_{j=p}^{n} a_j e^{s_0\lambda_j} \right| < \varepsilon$$

when $n \geq p \geq n_0$. Therefore, from (3.7) we get

$$\left| \sum_{n=p}^{q} a_n e^{s\lambda_n} \right| \leq \frac{\varepsilon |s_0 - s|}{\sigma_0 - \sigma} \sum_{n=p}^{q-1} \left[e^{(\sigma-\sigma_0)\lambda_n} - e^{(\sigma-\sigma_0)\lambda_{n+1}} \right] + \varepsilon e^{(\sigma-\sigma_0)\lambda_q} \leq$$

$$\leq \varepsilon \frac{|s_0 - s|}{\sigma_0 - \sigma} \left[e^{(\sigma-\sigma_0)\lambda_p} - e^{(\sigma-\sigma_0)\lambda_q} \right] + \varepsilon e^{(\sigma-\sigma_0)\lambda_q} \leq$$

$$\leq \varepsilon \frac{|s_0 - s|}{\sigma_0 - \sigma} e^{(\sigma-\sigma_0)\lambda_p} + \varepsilon e^{(\sigma-\sigma_0)\lambda_q} \leq \varepsilon \left(\frac{|s_0 - s|}{\sigma_0 - \sigma} + 1 \right) . \tag{3.8}$$

From the estimate (3.8) it immediately follows that the series (3.1) is convergent in each point of the half-plane $\{s : \operatorname{Re} s < \operatorname{Re} s_0\}$. If s belongs to the angle (3.6), then $|s_0 - s| / (\sigma_0 - \sigma) \leq 1/\cos\gamma < +\infty$ and from (1.8) it follows that

$$\left| \sum_{n=p}^{q} a_n e^{s\lambda_n} \right| \leq \varepsilon \left\{ \frac{1}{\cos\gamma} + 1 \right\},$$

that is, in the angle (3.6) the series (3.1) is uniformly convergent. The Theorem 3.1 is proved.

From the Theorem 3.1 it follows: if the series (1.1) converges at the point s_0, then its sum $F(s)$ is an analytical function in the half-plane $\{s:\operatorname{Re} s<\operatorname{Re} s_0\}$.

From this theorem it follows, that the series (3.1) either is convergent everywhere in C or it is divergent everywhere in C, or it is convergent in some half-plane $\{s:\operatorname{Re} s<\sigma_ç\}$ and divergent in the half-plane $\{s:\operatorname{Re} s>\sigma_ç\}$. In the last case, the number $\sigma_ç$ is called the abscissa of convergence of the series (3.1), the half-plane $\{s:\operatorname{Re} s<\sigma_ç\}$ is called the half-plane of convergence, the line $\{s:\operatorname{Re} s=\sigma_ç\}$ – the line of convergence. If the Dirichlet series converges in C, then we think that $\sigma_ç=+\infty$ and if it does not converge anywhere, then we think that $\sigma_ç=-\infty$.

Suppose that the series (3.1) at the point $s_0\in C$ is absolutely convergent, that is,

$$\sum_{n=1}^{\infty}|a_n|e^{\sigma_0\lambda_n}<+\infty,\quad \sigma_0=\operatorname{Re} s_0.$$

Then, if $\operatorname{Re} s\le\sigma_a$, then $|a_n|e^{s\lambda_n}\le|a_n|e^{\sigma_0\lambda_n}$ and, consequently, the series (3.1) is absolutely and uniformly convergent in the half-plane $\{s:\ \operatorname{Re} s<\sigma_0\}$. It follows that the series (3.1) is absolutely convergent everywhere in C, or at anyone point in C it is not absolutely convergent, or there exists a number $\sigma_a\in R$ such that the series converges absolutely in the half-plane $\{s:\ \operatorname{Re} s<\sigma_a\}$ and does not converge absolutely at every point of the half-plane $\{s:\ \operatorname{Re} s>\sigma_a\}$. This number σ_a is called the abscissa of the absolutely convergence of the Dirichlet series (3.1). If the Dirichlet series is absolutely convergent in C, then $\sigma_a=+\infty$, and if it is not absolutely convergent anywhere, then $\sigma_a=-\infty$.

As noted above, not always $\sigma_a=\sigma_ç$. For example, for the series

$$1+\sum_{n=1}^{\infty}(-1)^n\exp(s\ln n) \tag{3.9}$$

we have $\sigma_a = 1$ and $\sigma_ç$ =0. There are the Dirichlet series for which $\sigma_a = -\infty$ and $\sigma_ç = +\infty$. For examle,

$$1 + \sum_{n=3}^{\infty} (-1)^n \frac{1}{n} \exp\left(s\sqrt{\ln\ln n}\right) . \tag{3.10}$$

In fact, let $s = \sigma > 0$. Then $\frac{1}{n}\exp\left(\sigma\sqrt{\ln\ln n}\right) \to 0 \, (n \to \infty)$. To apply the Leibniz sign, it is enough to show that the inequality is true for sufficiently large n

$$\frac{1}{n}\exp\left(\sigma\sqrt{\ln\ln n}\right) \to \frac{1}{n-1}\exp\left(\sigma\sqrt{\ln\ln(n-1)}\right),$$

and it's true, because

$$\sigma\left(\sqrt{\ln\ln n} - \sqrt{\ln\ln(n-1)}\right) = \frac{(1+o(1))\sigma}{2n\ln n\sqrt{\ln\ln n}}, \, n \to \infty,$$

$$\ln n - \ln(n-1) = (1+o(1))\frac{1}{n}, \, n \to \infty .$$

So the series (3.10) converges at each point $s = \sigma > 0$ and by the Theorem3.1 it converges everywhere in C. However

$$\sum_{n=3}^{\infty} \frac{1}{n}\exp\left(\sigma\sqrt{\ln\ln n}\right) = +\infty$$

for each $\sigma \in R$, that is $\sigma_a = +\infty$.

Note that $\sigma_à \le \sigma_ç$. If $\sigma_a > -\infty$, then for arbitrary $\varepsilon > 0$ in each half-plane $\{s : \operatorname{Re} s < \sigma_a - \varepsilon\}$ the series (1.1) is uniformly convergent. The upper bound of the numbers α, such that the series (3.1) is uniformly convergent in the half-plane $\{s : \operatorname{Re} s \le \alpha\}$ is called the abscissa $\sigma_ð$ of uniformly convergence . Then

$$\sigma_à \le \sigma_ð \le \sigma_ç .$$

Let's determine the abscissa of convergence. Denote

$$\alpha_0 \overset{def}{=} \varliminf_{n\to\infty} \frac{1}{\lambda_n} \ln \frac{1}{|a_n|} = \varliminf_{n\to\infty} \frac{-\ln|a_n|}{\lambda_n}, \tag{3.11}$$

and for $\gamma > 0$ and $\delta \in R$ let

$$h(\gamma,\delta) \overset{def}{=} \varliminf_{n\to\infty} \frac{(\gamma-1)\ln|a_n| + \delta\lambda_n}{\ln n}. \tag{3.12}$$

Theorem 3.2. *For each Dirichlet series (3.1)* $\sigma_{à} \le \sigma_{ç}, \sigma_{à} \le \alpha_0$. *If* $h(\gamma,\delta) > 1$, *then* $\sigma_{à} \ge \gamma\sigma_{ç} - \delta$ *and* $\sigma_{à} \ge \gamma\alpha_0 - \delta$.

Proof. The inequality $\sigma_{à} \le \sigma_{ç}$ is obvious. Let $\alpha_0 < +\infty$. Then for each $\alpha > \alpha_0$ there exists the increasing sequence of natural numbers (n_k), such that

$$|a_{n_k}| \ge \exp(-\alpha\lambda_{n_k}),$$

i.e.

$$|a_{n_k}| \exp(\sigma\lambda_{n_k}) \ge \exp\{(-\alpha+\sigma)\lambda_{n_k}\} \ge 1$$

for all $\sigma \ge \alpha$. Hence, due to the arbitrariness of α, it follows that the series (3.1) converges at each point $s = \sigma > \alpha_0$ and, hence, $\sigma_{à} \le \alpha_0$. For $\alpha_0 = +\infty$ this inequality is obvious.

If $\alpha_0 > -\infty$, then for each $\sigma < \gamma\alpha_0 - \delta$ we have

$$\varliminf_{n\to\infty} \frac{1}{\lambda_n} \ln \frac{1}{a_n} > \frac{\sigma+\delta}{\gamma},$$

i.e.

$$\ln|a_n| + \frac{\sigma+\delta}{\gamma}\lambda_n = \lambda_n\left(-\frac{1}{\lambda_n}\ln\frac{1}{a_n} + \frac{\sigma+\delta}{\gamma}\right) < 0$$

for all sufficiently large n. Therefore, for such n the next inequality

$$|a_n|\exp(\sigma\lambda_n) = \exp\left\{-\left((\gamma-1)\ln|a_n| + \delta\lambda_n\right) + \gamma\left(\ln|a_n| + \frac{\sigma+\delta}{\gamma}\lambda_n\right)\right\} \le$$

$$\le \exp\left\{-\frac{(\gamma-1)\ln|a_n| + \delta\lambda_n}{\ln n}\ln n\right\} < \exp\{-(h(\gamma,\delta)-\varepsilon)\ln n\} \tag{3.13}$$

is hold for each $\varepsilon \in (0; h(\gamma,\delta)-1)$ and all $n \ge n_0(\varepsilon)$. Since $h(\gamma,\delta)-\varepsilon > 1$, then it follows that the series (3.1) is absolutely convergent at the point $s=\sigma$, that is $\sigma_{à} \ge \gamma\alpha_0 - \delta$. For $\alpha_0 = -\infty$ this inequality is obvious.

Let, finally, $\sigma_{ç} > -\infty$ and $\eta \in (-\infty; \sigma_{ç})$. Since the series (3.1) converges at the point η, then there exists $M > 0$, such that for all $n \ge 0$

$$|a_n| \exp(\eta\lambda_n) \le M ,$$

i.e.

$$|a_n| \exp\{(\gamma\eta - \delta)\lambda_n\} = \left(|a_n| \exp\{\eta\lambda_n\}\right)^{\gamma} |a_n|^{1-\gamma} \exp\{-\delta\lambda_n\} \le$$

$$\le M^{\gamma} \exp\left\{-\frac{(\gamma-1)\ln|a_n| + \delta\lambda_n}{\ln n} \ln n\right\} < \exp\left\{-\left(h(\gamma,\delta)-\varepsilon\right)\ln n\right\}$$

for each $\varepsilon \in (0; h(\gamma,\delta)-1)$ and all $n \ge n_0(\varepsilon)$. Then it follows, as from (1.13), that $\sigma_a \ge \gamma\eta - \delta$ and due to the arbitrariness of η, $\sigma_{à} \ge \gamma\sigma_{ç} - \delta$. For $\sigma_{ç} = -\infty$ this inequality is obvious. The theorem is proved.

Let's denote

$$\tau_0 \overset{def}{=} \varlimsup_{n\to\infty} \frac{1}{\lambda_n} \ln n . \tag{3.14}$$

Corollary 3.1. *For each Dirichlet series (3.1) the inequalities* $\sigma_{à} \le \sigma_{ç} \le \sigma_{à} + \tau_0$ *and* $\sigma_{à} \le \alpha_0 \le \sigma_{à} + \tau_0$ *are hold.*

In fact, let in the Theorem 3.1 $\gamma = 1$ and $\delta = \tau_0 + \varepsilon, \varepsilon > 0$. Then

$$h(\gamma,\delta) = \varliminf_{n\to\infty} \frac{(\tau_0+\varepsilon)\lambda_n}{\ln n} = \frac{\tau_0+\varepsilon}{\tau_0} > 1$$

and, therefore, the estimates $\sigma_{à} \ge \alpha_0 - \tau_0 - \varepsilon$ and $\sigma_{à} \ge \sigma_{ç} - \tau_0 - \varepsilon$ are correct, that is, taking into account arbitrariness of ε, we have $\sigma_{à} \ge \alpha_0 - \tau_0$ and $\sigma_{à} \ge \sigma_{ç} - \tau_0$.

The examples of series (3.9) and (3.10) indicate that these estimates are exact. For the first of these series we get that $\tau_0 = 1, \alpha_0 = \sigma_ç = 0, \sigma_à = -1$ and for the second series we have $\tau_0 = \alpha_0 = \sigma_ç = +\infty, \sigma_à = -\infty$.

Corollary 3.2. *If*

$$\underline{\lim}_{n\to\infty} \frac{\ln|a_n|}{\ln n} = h_0 > 0,$$

then $\sigma_à \ge \alpha_0 (h_0 + 1)/h_0$ *and* $\sigma_à \ge \sigma_ç (h_0 + 1)/h_0$.

In fact, let in the Theorem 3.1 $\delta = 0, \gamma = 1 + \varepsilon + 1/h_0$. Then

$$h(\gamma, \delta) = \underline{\lim}_{n\to\infty} \left(\frac{1}{h_0} + \varepsilon \right) \frac{\ln|a_n|}{\ln n} \ge 1 + \varepsilon h_0 > 1$$

and, hence, $\sigma_à \ge (1 + \varepsilon + 1/h_0)\sigma_ç$ and $\sigma_à \ge (1 + \varepsilon + 1/h_0)\alpha_0$, due to the arbitrariness of ε, we get the necessary inequalities.

Example of a series

$$1 + \sum_{n=3}^{\infty} (-1)^n n \exp(s \ln n)$$

indicates the exactness of the 2 mentioned estimates in the corollary, where $h_0 = 1, \alpha_0 = \sigma_ç = -1, \sigma_à = -2$.

Corollory 3. 3. *If*

$$\overline{\lim_{n\to\infty}} \frac{\ln n}{\ln(1/|a_n|)} = h_0 \in [0;1),$$

then and $\sigma_à \ge \sigma_ç (1 - h_0)$ *and* $\sigma_à \ge \alpha_0 (1 - h_0)$.

In fact, if in the Theorem 1.3 we take $\delta = 0$ and $\gamma = 1 - h_0 - \varepsilon, 0 < \varepsilon < 1 - h_0$, then

$$h(\gamma, \delta) = \underline{\lim}_{n\to\infty} \frac{(h_0 + \varepsilon)\ln(1/|a_n|)}{\ln n} = \frac{h_0 + \varepsilon}{h_0} > 1$$

and, consequently, $\sigma_à \ge \sigma_ç (1 - h_0 - \varepsilon)$ and $\sigma_à \ge \alpha_0 (1 - h_0 - \varepsilon)$, then due to the arbitrariness of ε we get the necessary inequalities.

The example of the series

$$1+\sum_{n=3}^{\infty}(-1)^n \frac{1}{n^2}\exp(s\ln n)$$

indicates the exactness of the 2 mentioned estimates in the corollary. Here $h_0 = 1/2, \alpha_0 = \sigma_ç = 2, \sigma_a = 1.$

Using the Corollaries of 3.1-3.3, we will prove such a theorem.

Theorem 3.3. *Let* $\tau_0 = 0$ *or* $\ln n = o(\ln|a_n|), n \to \infty$. *Then* $\sigma_à = \sigma_ç = \alpha_0$.

Proof. If $\tau_0 = 0$, then the conclusion of the theorem follows from the Corollary 3.1. If $\ln n = o(\ln|a_n|), n \to \infty$, then either 1) $\ln|a_n| / \ln n \to +\infty \ (n \to \infty)$, or 2) $\ln|1/a_n| / \ln n \to +\infty \ (n \to \infty)$, or $\{a_n\} = \{a'_n\} \bigcup \{a''_n\}$, where $\{a'_n\}$ satisfies the condition 1), and $\{a''_n\}$ satisfies the condition 2). In the first two cases, the correctness of the theorem follows from the corollaries 3.2 and 3.3. In the third case, we'll write the series (3.1) in the form of the sum

$$a_0 + \sum_{n=1}^{\infty} a'_n e^{s\lambda'_n} + \sum_{n=1}^{\infty} a''_n e^{s\lambda''_n}, \tag{3.15}$$

where λ'_n and λ''_n correspond to the coefficients a'_n and a''_n. By the collroraries 3.2 and 3.3, we have

$$\sigma'_ç = \sigma'_à = \alpha'_0 = \varlimsup_{n\to\infty} \frac{1}{\lambda'_n}\ln\frac{1}{a'_n}, \quad \sigma''_ç = \sigma''_à = \alpha''_0 = \varlimsup_{n\to\infty} \frac{1}{\lambda''_n}\ln\frac{1}{a''_n},$$

where $\sigma'_ç, \sigma'_à$ and $\sigma''_ç, \sigma''_à$ are the abscissas of convergence and the absolute convergence of the corresponding series from (3.15). Note also, that

$$\sigma_à = \min\{\sigma'_a, \sigma''_a\}, \text{ a } \sigma_ç = \min\{\sigma'_ç, \sigma''_ç\}.$$

It follows that

$$\alpha_0 \le \min\{\alpha'_0, \alpha''_0\} = \min\{\sigma'_a, \sigma''_a\} = \sigma_a \le \alpha_0$$

and

$$\sigma_ç \le \min\{\sigma'_a, \sigma''_a\} = \sigma_a \le \sigma_ç.$$

The Theorem 3.3 is completely proved.

Analogues of Cauchy's inequality

If the function f is given by the power series (3.2) and is analytical in the circle $\{z:|z|<R\}$, then, as it is known from the general course of Complex Analysis, $|a_n|r^n \le M_f(r)$ for all $n\ge 0$ and $0\le r<R$, $M_f(r)=\max\{|f(z)|:|z|=r\}$.

Let the Dirichlet series (3.1) has an absolute abscissa $\sigma_a\in(-\infty;+\infty]$ of convergence. For $\sigma<\sigma_a$ we denote

$$M(\sigma)=M(\sigma,F)=\sup\{|F(\sigma+it)|:t\in R\}.$$

Theorem 3.4. *For all* $n\in Z_+$ *and* $\sigma<\sigma_a$ *the inequality*

$$|a_n|\exp(\sigma\lambda_n)\le M(\sigma,F)$$

is hold.

Proof. This inequality easily follows from the such true equality for all $n\in Z_+$ *and* $\sigma<\sigma_a$:

$$a_n e^{\sigma\lambda_n} = \lim_{T\to\infty}\frac{1}{T}\int_{-T}^{T} F(\sigma+it)e^{-it\lambda_n}dt. \tag{3.16}$$

Let's prove this equality. We have

$$\frac{1}{2T}\int_{-T}^{T}F(\sigma+it)e^{-(\sigma+it)\lambda_n}dt=\frac{1}{2T}\int_{-T}^{T}\sum_{m=0}^{\infty}a_m e^{(\sigma+it)\lambda_m}e^{-(\sigma+it)\lambda_n}dt=$$

$$=\frac{1}{2T}\int_{-T}^{T}\left\{\sum_{m=0}^{n-1}a_m e^{(\sigma+it)(\lambda_m-\lambda_n)}dt+a_n+\sum_{m=n+1}^{\infty}a_m e^{(\sigma+it)(\lambda_m-\lambda_n)}dt\right\}.$$

The last series is uniformly convergent relatively to $t\in R$, it can be integrated term by term and therefore, using the relations

$$\frac{1}{2T}\int_{-T}^{T}e^{x(\sigma+it)}dt=\frac{1}{ixT}\left(e^{ixT}-e^{-ixT}\right)\to 0,\quad (T\to+\infty),\quad x\ne 0,$$

we get

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}F(\sigma+it)e^{-i(\sigma+it)\lambda_n}dt=a_n,$$

That is, we obtain the equality (3.16).

Let’s denote

$$B(\sigma)=B(\sigma,F)=\sup\{\operatorname{Re}F(\sigma+it):t\in R\},\ \sigma<\sigma_a\ .$$

Theorem 3.5. *For all* $n\in N$ *and* $\sigma<\sigma_a$ *the inequality*

$$|a_n|\exp(\sigma\lambda_n)\le 2B(\sigma,F)-2\operatorname{Re}a_0$$

is true.

Proof. Consider the Dirichlet series

$$F_1(s)=B(\sigma_0)-a_0-\sum_{n=1}^{\infty}a_ne^{s\lambda_n}=\sum_{n=0}^{\infty}b_ne^{s\lambda_n},$$

where σ_0 is an arbitrary number, $\sigma_0<\sigma_a$. Then for each $\sigma<\sigma_a$

$$b_ne^{\sigma\lambda_n}=\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}F_1(\sigma+it)e^{-it\lambda_n}dt=$$

$$=\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}P(\sigma+it)e^{-it\lambda_n}dt+i\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}Q(\sigma+it)e^{-it\lambda_n}dt,$$

where $P=\operatorname{Re}F_1$ and $Q=\operatorname{Im}F_1$. On the other hand, for $n\ge1$ and $\sigma<\sigma_a$

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}P(\sigma+it)e^{-it\lambda_n}dt+i\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}Q(\sigma+it)e^{-it\lambda_n}dt=$$

$$=\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\sum_{k=0}^{T}b_ke^{\sigma\lambda_k}e^{it(\lambda_n+\lambda_k)}dt=$$

$$=\lim_{T\to+\infty}\sum_{k=0}^{\infty}b_ke^{\sigma\lambda_k}\frac{1}{iT(\lambda_n+\lambda_k)}\left(e^{iT(\lambda_n+\lambda_k)}-e^{iT(\lambda_n+\lambda_k)}\right)=0,$$

whence

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}P(\sigma+it)e^{-it\lambda_n}dt-i\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}Q(\sigma+it)e^{-it\lambda_n}dt=0.$$

Therefore, for all $\sigma<\sigma_a$

$$b_n e^{\sigma\lambda_n}=2\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}P(\sigma+it)e^{-it\lambda_n}dt.$$

Since $\operatorname{Re}F(\sigma_0+it)\le B(\sigma_0,F)$, then $P(\sigma_0+it)\ge 0$. So,

$$b_n e^{\sigma_0\lambda_n}\le 2\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}P(\sigma_0+it)dt=$$

$$=2\operatorname{Re}b_0=2\operatorname{Re}\left(B(\sigma_0,F)-a_0\right)=2B(\sigma_0,F)-2\operatorname{Re}a_0.$$

Let's return to (3.2) and denote

$$B_f(r)=\max\left\{\operatorname{Re}f\left(re^{i\theta}\right):\theta\in[0;2\pi]\right\},\ 0\le r<R.$$

If $z=e^s$ (consequently, $r=e^{\sigma}$), then by the Theorem 3.5 we easily obtain the inequality

$$|a_n|r^n\le 2\left(B_f(r)-\operatorname{Re}a_0\right)\quad (n\ge 1,\ 0\le r<R).$$

Three-lines theorem

Firstly, we will prove one theorem of Phragmen-Lindelöf type.

Theorem 3.6. *If the function F is analytical and bounded in the closed strip, $\{s:\sigma_1\le\operatorname{Re}s\le\sigma_2\}$ and the inequalities $\left|F\left(\sigma_j+it\right)\right|\le G\in(0;+\infty)$ are hold on the lines $\left\{s:\operatorname{Re}s\le\sigma_j\right\}$, $j=1,2$, then $F(s)\le G$ for all $s,\sigma_1\le\operatorname{Re}s\le\sigma_2$.*

Proof. Let's consider the analytical function in this strip

$$g(s)=F(s)e^{\varepsilon s^2},\ \varepsilon>0.$$

Then

$$\left|g(s)\right| = \left|F(s)\exp\left\{\varepsilon\left(\sigma^2 - t^2 + 2it\sigma\right)\right\}\right| = \left|F(s)\right|\exp\left\{\varepsilon\left(\sigma^2 - t^2\right)\right\}, \qquad (3.17)$$

whence it follows that on the lines $\left\{s : \operatorname{Re} s \le \sigma_j\right\}\left(j = 1,2\right)$ the inequalities

$$\left|g(s)\right| \le G\exp\left\{\varepsilon\left(\sigma_j^2\right)\right\} \le G\exp\left\{\varepsilon\left(\sigma_0^2\right)\right\}, \quad \sigma_0^2 = \max\left\{\sigma_1^2;\sigma_2^2\right\}$$

are hold.

From (3.17) and the boundness of the function F it follows that

$$\left|g(s)\right| = O\left(\exp\left(-\varepsilon t^2\right)\right)$$

when $s \to \infty$ in the strip $\left\{s : \sigma_1 \le \operatorname{Re} s \le \sigma_2\right\}$. Therefore, there exists the number $t_0 > 0$, such that inequality

$$\left|g\left(\sigma + it\right)\right| \le G\exp\left\{\varepsilon\sigma_j^2\right\} \qquad (3.18)$$

is true when $\left|t\right| \ge t_0 > 0$ and $\sigma_1 \le \sigma \le \sigma_2$.

This inequality is correct at the edge of the rectangle $\left\{s = \sigma + it : \left|t\right| \le t_0, \sigma_1 \le \sigma \le \sigma_2\right\}$ and, therefore, in this rectangle and in the strip $\left\{s : \sigma_1 \le \operatorname{Re} s \le \sigma_2\right\}$. From (3.17) and (3.18) for each s from this strip we obtain

$$\left|F(s)\right| = \left|g(s)\right|e^{\varepsilon\left(t^2 - \sigma^2\right)} \le Ge^{\varepsilon\left(\sigma_0^2 + t^2 - \sigma^2\right)}.$$

Then, due to the arbitrariness of ε, we get the inequality $\left|F(s)\right| \le G$ for each $s, \sigma_1 \le \operatorname{Re} s \le \sigma_2$. The Theorem 3.6 is proved.

The Theorem 3.7 is called the three-lines theorem and it is the generalization of Hadamard's three-circles for the entire functions.

Theorem 3.7. *Let $\sigma_1 \le \sigma_2 \le \sigma_3$. If F is the analytical function in the strip $\left\{s : \sigma_1 \le \operatorname{Re} s \le \sigma_3\right\}$, is bounded in it and doesn't equal zero, then*

$$M\left(\sigma_2\right)^{\sigma_3 - \sigma_1} \le M\left(\sigma_1\right)^{\sigma_3 - \sigma_2} M\left(\sigma_3\right)^{\sigma_2 - \sigma_1}, \quad M\left(\sigma\right) = M\left(\sigma, F\right). \qquad (3.19)$$

Proof. Let $\alpha \in R$. Then the function $\varphi(s) = e^{\alpha s}F(s)$ is analytical and bounded in the strip $\left\{s : \sigma_1 \le \operatorname{Re} s \le \sigma_3\right\}$. At the edge of the strip the inequalities

$$\left|\varphi(s)\right| \le M\left(\sigma_j\right)e^{\alpha\sigma_j},\ j=1,3$$

are hold.

Therefore, by the Theorem 3.6 for $\sigma_1 \le \operatorname{Re} s \le \sigma_3$

$$\left|\varphi(s)\right| \le \max\left\{e^{\alpha\sigma_1}M\left(\sigma_1\right), e^{\alpha\sigma_3}M\left(\sigma_3\right)\right\},$$

i.e.

$$e^{\alpha\sigma_2}M\left(\sigma_2\right) \le \max\left\{e^{\alpha\sigma_1}M\left(\sigma_1\right), e^{\alpha\sigma_3}M\left(\sigma_3\right)\right\}.$$

Let's choose α so that

$$e^{\alpha\sigma_1}M\left(\sigma_1\right) = e^{\alpha\sigma_3}M\left(\sigma_3\right),$$

i.e.

$$\alpha = \frac{1}{\sigma_3-\sigma_1}\ln\frac{M\left(\sigma_1\right)}{M\left(\sigma_3\right)}.$$

Then from the last inequality we get

$$\left(\frac{M\left(\sigma_1\right)}{M\left(\sigma_3\right)}\right)^{\sigma_2/(\sigma_3-\sigma_1)} M\left(\sigma_2\right) \le \left(\frac{M\left(\sigma_1\right)}{M\left(\sigma_3\right)}\right)^{\sigma_2/(\sigma_3-\sigma_1)} M\left(\sigma_1\right),$$

whence we easily get the inequality (3.19).

Corollary 3.4. *Let the Dirichlet series (3.1) have the abscissa* $\sigma_a \in [-\infty;+\infty)$ *of convergence. Then the function* $\ln M(\sigma,F)$ *is convex on* $\left(-\infty;\sigma_a\right)$*, hence, it is continuous and has continuous derivative except, possibly, the countable number of points, in which there exists the one-sided derivatives, moreover the left-sided derivative dos not exceed the right-sided derivative.*

Indeed, since the series (3.1) converges absolutely in $\left\{s:\operatorname{Re} s \le \sigma_a\right\}$, then its sum F is analytical and bounded in each vertical strip from this half-plane, and by the Theorem 3.7 for each three numbers $-\infty<\sigma_1<\sigma_2<\sigma_3<\sigma_a$, the inequality (3.19) is true, from which the inequality

$$\left(\sigma_3-\sigma_1\right)\ln M\left(\sigma_2\right) \le \left(\sigma_3-\sigma_2\right)\ln M\left(\sigma_1\right)+\left(\sigma_2-\sigma_1\right)\ln M\left(\sigma_3\right)$$

follows. So, $\ln M(\sigma,F)$ is the convex function. The rest of the conclusions of the corollary follow from the convexity.

From the Theorem 3.7, using a replacement $z = e^s$ (hence, $r = e^{\sigma}$), we easily get the correctness of such a corollary.

Corollary 3.5 (Hadamar's three-circle theorem). *Let the power series (3.2) have a convergence radius* $R \in (0;+\infty]$. *Then for* $0 \le r_1 \le r_2 \le r_3 < R$

$$M_f(r_2)^{\ln r_3 - \ln r_1} \le M_f(r_1)^{\ln r_3 - \ln r_2} M_f(r_3)^{\ln r_2 - \ln r_1}.$$

It follows that the function $\ln M_f(r)$ is convex according the logarithm (logarithmically convex).

Maximum term and central index

Let the Dirichlet series (3.1) have the convergence abscissa $\sigma_a \in (-\infty;+\infty]$. Then for each $\sigma < \sigma_a$ we have $|a_n|\exp(\sigma\lambda_n) \to 0\,(n \to \infty)$ and therefore there exists the maximum term

$$\mu(\sigma) = \mu(\sigma,F) = \max\{|a_n|\exp(\sigma\lambda_n) : n \ge 0\}$$

of the series (3.1). The value

$$\nu(\sigma) = \nu(\sigma,F) = \max\{n : |a_n|\exp(\sigma\lambda_n) = \mu(\sigma)\}$$

is called the central index of the series (3.1), and $\lambda_{\nu(\sigma)}$ is its central indicator. The functions $\nu(\sigma)$ and $\lambda_{\nu(\sigma)}$ are piecawise-constant on $(-\infty;\sigma_a)$, moreover $\nu(\sigma)$ is only integers.

Remark. The definition shows that the maximum term and the central index can be defined for any sequence $(a_n \exp(\sigma\lambda_n))$ that tends to 0, when $n \to \infty$ for each $s, \operatorname{Re} s < A$, where A is a certain number from $(-\infty;+\infty]$.

Let $h > 0$ be an arbitrary number. Then

$$\mu(\sigma+h)=\left|a_{\nu(\sigma+h)}\right|\exp\left\{(\sigma+h)\lambda_{\nu(\sigma+h)}\right\}\geq\left|a_{\nu(\sigma)}\right|\exp\left\{(\sigma+h)\lambda_{\nu(\sigma)}\right\}=$$
$$=\left|a_{\nu(\sigma)}\right|\exp\left\{\sigma\lambda_{\nu(\sigma)}\right\}\exp\left\{h\lambda_{\nu(\sigma)}\right\}=\mu(\sigma)\exp\left\{h\lambda_{\nu(\sigma)}\right\},$$

i.e.

$$\ln\mu(\sigma+h)-\ln\mu(\sigma)\geq h\lambda_{\nu(\sigma)}. \tag{3.20}$$

Similarly,

$$\mu(\sigma)=\left|a_{\nu(\sigma)}\right|\exp\left\{\sigma\lambda_{\nu(\sigma)}\right\}\geq\left|a_{\nu(\sigma+h)}\right|\exp\left\{\sigma\lambda_{\nu(\sigma+h)}\right\}=$$
$$=\left|a_{\nu(\sigma+h)}\right|\exp\left\{(\sigma+h)\lambda_{\nu(\sigma+h)}\right\}\exp\left\{-h\lambda_{\nu(\sigma+h)}\right\}=\mu(\sigma+h)\exp\left\{-h\lambda_{\nu(\sigma+h)}\right\},$$

i.e.

$$\ln\mu(\sigma+h)-\ln\mu(\sigma)\leq h\lambda_{\nu(\sigma+h)}. \tag{3.21}$$

From (3.20) and (3.21) we obtain

$$\lambda_{\nu(\sigma)}\leq\frac{1}{h}\left(\ln\mu(\sigma+h)-\ln\mu(\sigma)\right)\leq\lambda_{\nu(\sigma+h)}.$$

It follows that the functions $\nu(\sigma), \lambda_{\nu(\sigma)}$ and $\ln\mu(\sigma)$ are non-decreasing. Our considerations are correct if $h<0$. Therefore, if (σ_1,σ_2) is the interval of constancy of the function $\nu(\sigma)$, then when $h\to 0$ we receive

$$\frac{d\ln\mu(\sigma)}{d\sigma}=\lambda_{\nu(\sigma)}.$$

Since at each finite interval the function $\lambda_{\nu(\sigma)}$ has a finite number of breakpoints, we get the equality

$$\ln\mu(\sigma)-\ln\mu(\sigma_0)=\int_{\sigma_0}^{\sigma}\lambda_{\nu(t)}dt \tag{3.22}$$

for all $-\infty<\sigma_0<\sigma<\sigma_a$. From (3.22) follows the convexity of the function $\ln\mu(\sigma)$

.

Suppose that the power series $f(z)=\sum_{n=0}^{\infty} a_n z^n$, $z=re^{i\varphi}$ has a convergence radius $R\in(0;+\infty)$. Then for each $r\in[0;R)$ there exists the maximum term $\mu_f(r)=\max\{|a_n|r^n : n\ge 0\}$ of this series. The value

$$\nu_f(r)=\max\{n:|a_n|r^n=\mu_f(r)\}$$

is called the central index of the series (3.2). If we make a replacement $z=e^s$ (hence, $r=e^{\sigma}$), then taking into account the correct equality in this case $\lambda_{\nu(\sigma)}=\nu(\sigma)$, from (3.22) we easily get the equality

$$\ln\mu_f(r)-\ln\mu_f(r_0)=\int_{r_0}^{r}\frac{\nu(t)}{t}dt,\quad 0\le r_0\le r<R.$$

Let's denote

$$\kappa_n=\kappa_n(F)=\frac{\ln|a_n|-\ln|a_{n+1}|}{\lambda_{n+1}-\lambda_n}.$$

Theorem 3.8. *If the sequence (κ_n) is non-decreasing, then $\mu(\kappa_n)=|a_n|\exp(\kappa_n\lambda_n)$ for all n. If $\kappa_{n-1}<\kappa_n$ for some $n\ge 1$, then $\nu(\sigma)=n$ and $\mu(\sigma)=|a_n|\exp(\sigma\lambda_n)$ for all $\sigma\in[\kappa_{n-1},\kappa_n)$ and this n.*

Proof. Let the sequence (κ_n) be non-decreasing. If $j<n$, then

$$|a_j|\exp(\kappa_n\lambda_j)=|a_n|\exp\left\{\sum_{m=j}^{n-1}\left((\ln|a_m|-\ln|a_{m+1}|)+\kappa_n\lambda_j\right)\right\}=$$

$$=|a_n|\exp\left\{\sum_{m=j}^{n-1}\left(\kappa_m(\lambda_{m+1}-\lambda_m)+\kappa_n\lambda_j\right)\right\}\le|a_n|\exp(\kappa_n\lambda_n),$$

and if $j>n$, then

$$|a_j|\exp(\kappa_n\lambda_j)=|a_n|\exp\left\{\sum_{m=n}^{j-1}\left((\ln|a_{m+1}|-\ln|a_m|)+\kappa_n\lambda_j\right)\right\}=$$

$$=|a_n|\exp\left\{-\sum_{m=n}^{j-1}\left(\kappa_m\left(\lambda_{m+1}-\lambda_m\right)+\kappa_n\lambda_j\right)\right\}\le|a_n|\exp\left(\kappa_n\lambda_n\right),$$

i.e. $\mu\left(\kappa_n\right)=|a_n|\exp\left(\kappa_n\lambda_n\right)$.

If $\kappa_{n-1}<\kappa_n$ for some n and $\sigma\in\left[\kappa_{n-1},\kappa_n\right)$, then similarly for $j<n$ we get

$$\frac{|a_j|\exp\left(\sigma\lambda_j\right)}{|a_n|\exp\left(\sigma\lambda_n\right)}=\exp\left\{\ln|a_j|-\ln|a_n|-\sigma\left(\lambda_n-\lambda_j\right)\right\}\le$$

$$\le\exp\left\{\kappa_{n-1}\left(\lambda_n-\lambda_j\right)-\sigma\left(\lambda_n-\lambda_j\right)\right\}=\exp\left\{-\left(\sigma-\kappa_{n-1}\right)\left(\lambda_n-\lambda_j\right)\right\}\le 1,$$

and for $j>n$

$$\frac{|a_j|\exp\left(\sigma\lambda_j\right)}{|a_n|\exp\left(\sigma\lambda_n\right)}\le\exp\left\{-\kappa_n\left(\lambda_j-\lambda_n\right)+\sigma\left(\lambda_j-\lambda_n\right)\right\}\le 1.$$

So, $\nu\left(\sigma\right)=n$ and $\mu\left(\sigma\right)=|a_n|\exp\left(\sigma\lambda_n\right)$ for all $\sigma\in\left[\kappa_{n-1},\kappa_n\right)$.

We will submit a geometric interpretation of the maximum term and the central index using Newton's diagram. Let's construct this diagram first for the simple case, where the Dirichlet series is reduced to an exponential polynomial

$$P\left(s\right)=a_0+\sum_{n=1}^{N}a_n\exp\left(s\lambda_n\right),\quad a_n\ne 0.$$

On the plane XOY we'll note the points $P_n=\left(\lambda_n,-\ln|a_n|\right)$. From the point P_0 we'll draw a vertically down ray and rotate it around the point P_0 (counterclockwise) until it touches the point with $\left\{P_n\right\}_{n\ge 1}$. Several such points may lie on the final placement of the ray. The farthest point from P_0 we'll denote by P_{n_1}, and the segment $\left[P_0;P_{n_1}\right]$ we'll denote by l_1. Now from the point P_{n_1} in the direction of l_1 we draw a ray again, rotate it around P_{n_1} to the touch with $\left\{P_n\right\}_{n>n_1}$ and the farthest point from P_{n_1} from the final placement of this ray denote by P_{n_2}, and denote the segment $\left[P_{n_1};P_{n_2}\right]$ by l_2. This process is finite and we will move on to the segment

$l_k = \left[P_{n_k}; P_N \right]$. Finally, from the point P_N we will draw a vertically up the ray l_{k+1}. As a result, we will get a convex broken L, the links of which are the segments $l_j, 1 \le j \le k$ and the ray l_{k+1}. It is called the Newton's diagram and has such a property that none of the points P_n lies below it.

Let's construct the Newton's diagram for the Dirichlet series. Among the coefficients a_n may occur such that equals zero, but we think that it does not reduce to the exponential polynomial. In fact, as the remark shows, we are constructing the Newton's diagram for a functional sequence $\left(a_n \exp\{s\lambda_n\}\right)$. We think that $a_0 \ne 0$.

Let α_0 be determined by the equality (3.14) and $\alpha_0 \in (-\infty; +\infty]$. By the Theorem 3.3 $\alpha_0 \ge \sigma_a$. Denote by φ_n the angel between rays $y = \dfrac{-\ln|a_n|}{\lambda_n} x$ and $y = 0, x \ge 0$. Then from (3.14) we receive

$$\varliminf_{n \to \infty} \varphi_n = arctg\, \alpha_0 .$$

As above, let $P_n = \left(\lambda_n, -\ln|a_n|\right)$ (in the case, when $a_n = 0$, we think that $-\ln|a_n| = +\infty$), and the ray l' with the beginning in P_0 rotates from the vertical down the placement counterclockwise until one of three situations arises:

1) the ray l' will touches any point from $\{P_n\}_{n\ge1}$, it has a finite number of points P_n, and all the rest points lie higher l';

2) on the ray l' the infinite number of points from $\{P_n\}_{n\ge1}$ lies, and under it there are no such points;

3) there are no points from $\{P_n\}_{n\ge1}$ on l', but during the further rotation under l' will be the infinite number of such points.

In the cases 2) and 3) the process of constructing the Newton's diagram is completed and this final placement of the ray l' we denote by L. It is clear that the angle of inclination of L is equal to $arctg\, \alpha_0$.

In the first case, the most distant from the point P_0 the point P_n from the final placement l' we'll denote by P_{n_1} and let $l_1 = \left[P_0; P_{n_1}\right]$. From the point P_{n_1} as a continuation of l_1, we will draw a ray l'' and turn it around P_{n_1} again until one of the situations described above arises. In the last two, the process is completed and by L we will denote the broken, consisting of l_1 and the final placement of the ray l''. In the first situation, the most distant from P_{n_1} point $P_n, n > n_1$, from the final placement l'', we'll denote by P_{n_2} and let it $l_2 = \left[P_{n_1}; P_{n_2}\right]$.

Continuing this process, we will construct a convex broken L, which will either consist of the infinite number of edges l_n, such that the angle ψ_n between l_n and the ray $y = 0(x \ge 0)$ increases to $arctg\,\alpha_0$, when n increases or consists of a finite number of edges l_n and a ray l^* that forms an angle $arctg\,\alpha_0$ with a ray $y = 0(x \ge 0)$.

Let $y = L(x)$ be the equation of this Newton's diagram L. Then the Dirichlet series

$$F_{ìí}(s) = |a_0| + \sum_{n=1}^{\infty} \exp\{-L(\lambda_n) + s\lambda_n\} \qquad (3.23)$$

is called the Newton`s majorant of the Dirichlet series. The series (3.11) can be formal. It is clear, that $\ln|a_n| \le -L(\lambda_n), n \ge 0$ and $\ln\left|a_{n_j}\right| = -L\left(\lambda_{n_j}\right), j \ge 0$.

Suppose at first that Newton's diagram consists of a finite number of edges and draw a line l with an angular coefficient $\sigma < \alpha_0$ so that it touches the broken L, but does not cross it. This line l in common can have with L either one point P_ν or the segment $[P_k; P_\nu], k = k(\sigma), \upsilon = \upsilon(\sigma)$. Let's show that is $\upsilon(\sigma)$ is the central index of the Dirichlet series and the series (3.23). Indeed, if $m < \upsilon(\sigma)$, then

$$\frac{\ln|a_m|-\ln\left|a_{\upsilon(\sigma)}\right|}{\lambda_{\upsilon(\sigma)}-\lambda_m}\le\frac{L\left(\lambda_{\upsilon(\sigma)}\right)-L(\lambda_m)}{\lambda_{\upsilon(\sigma)}-\lambda_m}\le\sigma,$$

whence it follows the inequality

$$|a_m|\exp(\sigma\lambda_m)\le\left|a_{\upsilon(\sigma)}\right|\exp\left(\sigma\lambda_{\upsilon(\sigma)}\right). \tag{3.24}$$

The same inequality is hold for Newton's majorant. If $m>\upsilon(\sigma)$, then

$$\frac{\ln\left|a_{\upsilon(\sigma)}\right|-\ln|a_m|}{\lambda_m-\lambda_{\upsilon(\sigma)}}\ge\frac{L(\lambda_m)-L\left(\lambda_{\upsilon(\sigma)}\right)}{\lambda_m-\lambda_{\upsilon(\sigma)}}\ge\sigma,$$

whence it again follows (3.24) and the corresponding analogue for Newton's majorants. From these inequalities we see that the term $\left|a_{\upsilon(\sigma)}\right|\exp\left(\sigma\lambda_{\upsilon(\sigma)}\right)$ is the maximum term of the Dirichlet series, and $\exp\left\{L\left(\lambda_{\upsilon(\sigma)}\right)+\sigma\lambda_{\upsilon(\sigma)}\right\}$ is the maximum term of the series (3.23). But $\ln\left|a_{\upsilon(\sigma)}\right|=-L\left(\lambda_{\upsilon(\sigma)}\right)$. Therefore, the series (3.1) and the series (3.24) have the same maximum terms and central indixes.

Since the equation of the line l with the angular coefficient σ, passing through the poin $P_{\upsilon(\sigma)}$t is

$$y=\sigma\left(x-\lambda_{\upsilon(\sigma)}\right)-\ln\left|a_{\upsilon(\sigma)}\right|,$$

then, substituting $x=0$, we see that $|\ln\mu(\sigma)|$ is a distance from the point of intersection l with the axis $x=0$ to 0.

Let's consider the case when Newton's diagram includes a ray $[P_N,\infty)$ with an angular coefficient α_0. Its equation looks is $y=\alpha_0x-(\ln|a_N|+\alpha_0\lambda_N)$. Therefore, $\ln|a_n|\le-L(\lambda_n)=\ln|a_N|+(\lambda_N-\lambda_n)\alpha_0$ for all $n\ge N$. It follows that for all $\sigma<\sigma_a(\le\alpha_0)$ and all $n\ge N$

$$|a_n|\exp(\sigma\lambda_n)=|a_N|\exp\{(\sigma-\alpha_0)\lambda_n+\alpha_0\lambda_N\}\le|a_N|\exp(\alpha_0\lambda_N),$$

i. e. the maximum term is a bounded function.

This cannot be said about the central index. For the series

$$\sum_{n=0}^{\infty} \exp(sn)$$

we get $\sigma_a = \alpha_0 = 0, \mu(\sigma) = 1$ and $\nu(\sigma) = 0$ for all $\sigma < 0$, and for the series

$$1 + \sum_{n=1}^{\infty} \exp\left(-\frac{1}{n} + sn\right)$$

we have $\sigma_a = \alpha_0 = 0, \mu(\sigma) = 1$ for all $\sigma < 0$, but

$$\nu(\sigma) = \frac{1}{\sqrt{\sigma}} + O(1) \to +\infty \quad (\sigma \to 0).$$

In fact, the function $g(x) = -\frac{1}{x} + \sigma x, x \ge 1$ reaches its maximum at the point $x(\sigma) = 1/\sqrt{|\sigma|}$, $|\nu(\sigma) - x(\sigma)| \le 1$.

Estimates of the maximum modulus through the maximum term

From the Cauchy's inequality (Theorem 3.4) it follows that $\mu(\sigma, F) \le M(\sigma, F)$. Therefore, we need the estimates $M(\sigma, F)$ through $\mu(\sigma, F)$ from above.

Theorem 3.9. *Let* $\sigma_a = A, \gamma > 0$ *be an arbitrary number, and* $\delta = A(\gamma - 1)$, *when* $A < +\infty$, *and* $\delta \ge 0$ *be an arbitrary number, when* $A = +\infty$. *If* $h(\gamma, \delta) > 1$ *(see 3.15), then for all* $\sigma \in (\sigma_0; A)$

$$M(\sigma) \le K\mu\left(\frac{\sigma + \delta}{\gamma}\right)^{\gamma}, \quad K \equiv const \ . \tag{3.25}$$

Proof. We can assume that all $a_n \ne 0$ and for $n \ge 1$ denote

$$r_n = \frac{1}{\lambda_n} \ln \frac{1}{|a_n|}.$$

Then by the Theorem 3.3 $\varliminf\limits_{n\to\infty} r_n = \alpha_0 \ge A$. We accept $q(x) = \gamma x - \delta$, if $x \le A \le +\infty$, and $q(x) = \gamma A - \delta = A$, if $x \ge A$ in the case $A < +\infty$. Then

$$q^{-1}(\sigma) = \frac{\sigma+\delta}{\gamma} \le \frac{A+\delta}{\gamma} = A$$

for all $\sigma \in (-\infty; A)$ and

$$M(\sigma, F) \le \sum_{n=0}^{\infty} |a_n| \exp(\sigma\lambda_n) =$$

$$= \left(\sum_{r_n < q^{-1}(\sigma)} + \sum_{q^{-1}(\sigma) \le r_n < A} + \sum_{r_n \ge A} \right) |a_n| \exp(\sigma\lambda_n) + |a_0| \ . \qquad (3.26)$$

(if $A = +\infty$, then there is no third term in (3.26)).

If $r_n < q^{-1}(\sigma) = \dfrac{\sigma+\delta}{\gamma}$, then

$$e^{\sigma\lambda_n} = \left(|a_n| \exp\left\{ \lambda_n \frac{\sigma+\delta}{\gamma} \right\} \right)^{\gamma} \exp\{-\delta\lambda_n - \gamma \ln|a_n|\} \le$$

$$\le \mu\left(\frac{\sigma+\delta}{\gamma}\right)^{\gamma} \exp\{\lambda_n(\gamma r_n - \delta)\} = \mu\left(\frac{\sigma+\delta}{\gamma}\right)^{\gamma} \exp\{\lambda_n q(r_n)\}.$$

If $q^{-1}(\sigma) \le r_n < A$, then $\exp(\sigma\lambda_n) \le \exp(\lambda_n q(r_n))$. Finally, if $A < +\infty$, then $\exp(\sigma\lambda_n) \le \exp(A\lambda_n) = \exp(\lambda_n q(A)) = \exp(\lambda_n q(r_n))$. Therefore, from the inequality (3.26) we obtain

$$M(\sigma, F) \le \mu\left(\frac{\sigma+\delta}{\gamma}\right)^{\gamma} \sum_{r_n < q^{-1}(\sigma)} |a_n| \exp\{\lambda_n q(r_n)\} +$$

$$\sum_{q^{-1}(\sigma) \le r_n < A} |a_n| \exp\{\lambda_n q(r_n)\} + \sum_{r_n \ge A} |a_n| \exp\{\lambda_n q(r_n)\} + |a_0|.$$

Due to the condition $h(\gamma, \delta) > 1$,

$$\sum_{n=1}^{\infty} |a_n| \exp\{\lambda_n q(r_n)\} = \sum_{n=1}^{\infty} |a_n| \exp\left\{ \lambda_n \left(\frac{\gamma}{\lambda_n} \ln \frac{1}{\lambda_n} \right) \right\} =$$

$$=\sum_{n=1}^{\infty}\exp\left\{-\left((\gamma-1)\ln|a_n|+\delta\lambda_n\right)\right\}=\sum_{n=1}^{\infty}\exp\left\{-\frac{(\gamma-1)\ln|a_n|+\delta\lambda_n}{\ln n}\ln n\right\}<+\infty.$$

The Theorem 3.8. is proved.

Corollary 3.6. *If* $\sigma_a=+\infty$ *and* $\overline{\lim_{n\to\infty}}\frac{\ln n}{\lambda_n}=\tau_0<+\infty$*, then for each* $\varepsilon>0$ *and all* $\sigma\geq\sigma_0(\varepsilon)$

$$M(\sigma)\leq\mu(\sigma+\tau_0+\varepsilon). \tag{3.27}$$

Indeed, let's take $\delta=\tau_0+\varepsilon/2$ and $\gamma=1$. Then by the Theorem 3.9,

$$M(\sigma)\leq K\mu\left(\sigma+\tau_0+\frac{\varepsilon}{2}\right),\quad K\equiv const>0 \tag{3.28}$$

for all $\sigma\geq\sigma_1(\varepsilon)$. But

$$\ln\mu(\sigma+\tau_0+\varepsilon)-\ln\mu\left(\sigma+\tau_0+\frac{\varepsilon}{2}\right)=\int\limits_{\sigma+\tau_0+\varepsilon/2}^{\sigma+\tau_0+\varepsilon}\lambda_{\nu(t)}dt\geq\frac{\varepsilon}{2}\lambda_{\nu(\sigma)}\geq K$$

for all $\sigma\geq\sigma_2(\varepsilon)$, so from (3.28) we get (3.27).

Corollary 3.7. *If* $\sigma_a=+\infty$ *and*

$$\overline{\lim_{n\to\infty}}\frac{\ln n}{-\ln|a_n|}\leq h_0<1, \tag{3.29}$$

then for any $\varepsilon\in(0,1-h_0)$ *and all* $\sigma\geq\sigma_0(\varepsilon)$

$$M(\sigma)\leq\mu\left(\frac{\sigma}{1-h_0-\varepsilon}\right)^{1-h_0}. \tag{3.30}$$

In fact, if we choose $\delta=0$ and $\gamma=1-h_0-\varepsilon$, then by the Theorem 3.2

$$M(\sigma)\leq K\mu\left(\frac{\sigma}{1-h_0-\varepsilon}\right)^{1-h_0-\varepsilon},\quad K=K(\varepsilon)\equiv const>0.$$

Since $\mu(\sigma)\to+\infty\,(\sigma\to+\infty)$ for the entire Dirichlet series, then it follows (3.30).

Corollary 3.8. *Let* $\sigma_a = +\infty, \Phi \in \Omega(+\infty)$ *and* $\ln \mu(\sigma, F) \le \Phi(\sigma)$ *for all* $\sigma \in R$*. Let the function* φ *be inverse to* Φ' *and the function* Ψ *be associated with* Φ_b *by Newton. If*

$$\varlimsup_{n\to\infty} \frac{\ln n}{\lambda_n \Psi(\varphi(\lambda_n))} \le \beta_0 < 1, \qquad (3.31)$$

then for each $\varepsilon \in (0; 1-\beta_0)$ *and all* $\sigma \ge \sigma_0(\varepsilon)$

$$M(\sigma) \le \mu\left(\frac{\sigma}{1-\beta_0-\varepsilon}\right)^{1-\beta_0}. \qquad (3.32)$$

In fact, $\ln|a_n| \le -\lambda_n \Psi(\varphi(\lambda_n))$ and from (3.31) follows (3.29), when $h_0 = \beta_0$. Therefore, from (3.30) follows (3.32).

Consider the case, when $\sigma_a = 0$ (the general case $\sigma_a \in R$ is reduced to the case $\sigma_a = 0$ of the replacement with s by $s - \sigma_a$). In this case the maximum term can be the bounded function on $(-\infty; 0)$.

In order that $\mu(\sigma, F) \le K$ for all $\sigma < 0$, it is necessary and sufficient that $|a_n| \le K$ for all $n \ge 0$. Indeed, if $\mu(\sigma, F) \le K$, then $|a_n| \le K \exp\{-\sigma\lambda_n\}$ for all $\sigma < 0$ and $n \ge 0$. Tending $\sigma \uparrow 0$, we get that $|a_n| \le K$. Conversely, if $|a_n| \le K$, then $|a_n| \exp\{\sigma\lambda_n\} \le K$ for all $\sigma < 0$ and $n \ge 0$, i. e. $\mu(\sigma, F) \le K$.

In the case, when $|a_n| \le K$ for all $n \ge 0$, the estimate $\mu(\sigma, F) \le K$ is as follows:

$$M(\sigma, F) \le \sum_{n=0}^{\infty} |a_n| \exp(\sigma\lambda_n) \le K \sum_{n=0}^{\infty} \exp(\sigma\lambda_n)$$

and depends only on the density of indicators. Therefore, we consider that

$$\varlimsup_{n\to\infty} |a_n| = +\infty \ . \qquad (3.33)$$

On the other hand, if $\sigma_a = 0$, then in the Theorem 3.9 we can take only $\delta = 0$, and therefore, if $\gamma > 0$ is an arbitrary number and

$$\underline{\lim_{n\to\infty}}\frac{(\gamma-1)\ln|a_n|}{\ln n}=h(\gamma,0)>1, \qquad (3.34)$$

then for all $\sigma<0$

$$M(\sigma,F)\le K\mu\left(\frac{\sigma}{\gamma}\right)^{\gamma}, \quad K\equiv const>0\ . \qquad (3.35)$$

If $0<\gamma<1$, then (3.34) is hold only, when

$$\overline{\lim_{n\to\infty}}\frac{\ln|a_n|}{\ln n}<-\frac{1}{1-\gamma},$$

and this is possible from the point of view of (3.33). The condition (3.34) isn't hold in the case $\gamma=1$. Therefore, there remains the case $\gamma>1$. In order that the condition (3.34) is hold, it is necessary that

$$\underline{\lim_{n\to\infty}}\frac{\ln|a_n|}{\ln n}\ge\frac{1}{h_0}>0. \qquad (3.36)$$

Corollary 3.9. *If* $\sigma_a=0$ *and the condition (3.36) is hold, then for each* $\varepsilon>0$ *and all* $\sigma\in(\sigma_0(\varepsilon),0)$

$$M(\sigma)\le\mu\left(\frac{\sigma}{1+h_0+\varepsilon}\right)^{1+h_0+\varepsilon}. \qquad (3.37)$$

In fact, if we take $1+h_0+\varepsilon/2$, then from (3.36) we will have $h(\lambda,0)>1$, that is, (3.34) is hold and hence, (3.35) is hold, when $\gamma=1+h_0+\varepsilon/2$. Since due to (3.33) $\mu(\sigma,F)\uparrow+\infty(\sigma\uparrow 0)$ takes place, then from (3.35) follows (3.37).

Theorem 3.10. *Let* $\sigma_a=0$ *and the sequence satisfies the condition* (λ_n)

$$\overline{\lim_{n\to\infty}}\frac{\ln n}{\lambda_n\gamma(\lambda_n)}\le h_0<+\infty, \qquad (3.38)$$

where γ *is the positive continuous decreasing to 0 on* $[0;+\infty)$ *function, such that the function* $t\gamma(t)\uparrow+\infty\ (t\to+\infty)$. *Then for each* $\varepsilon>0$ *there exists the constant* $K(\varepsilon)>0$ *, such that for all* $\sigma<0$

$$M(\sigma) \le \mu\left(\frac{\sigma}{1+\varepsilon}\right)\left(\exp\left\{\frac{\varepsilon|\sigma|}{1+\varepsilon}\gamma^{-1}\left(\frac{\varepsilon|\sigma|}{(1+\varepsilon)^2\left(h_0+\varepsilon^2\right)}\right)\right\}\right)+K(\varepsilon). \qquad (3.39)$$

Proof. Let $n(t)=\sum_{\lambda_n \le t} 1$ is the counting function of the sequence (λ_n). Then the condition (3.38) is equivalent to the condition

$$\overline{\lim_{t\to+\infty}} \frac{\ln n(t)}{t\gamma(t)} \le h_0 < +\infty$$

and, hence, for each $\varepsilon > 0$, when $t \ge t_0 = t_0(\varepsilon)$ the inequality $\ln n(t) \le ht\gamma(t)$ is correct, where $h = h_0 + \varepsilon^2$. Taking the Stiltiès integral and integrating it by parts, we obtain

$$\frac{M(\sigma)}{\mu\left(\frac{\sigma}{1+\varepsilon}\right)} \le \sum_{n=0}^{\infty} \frac{|a_n|\exp\left(\lambda_n \frac{\sigma}{1+\varepsilon}\right)}{\mu\left(\frac{\sigma}{1+\varepsilon}\right)} \exp\left\{-\lambda_n \frac{\varepsilon|\sigma|}{1+\varepsilon}\right\} \le$$

$$\le \sum_{n=0}^{\infty} \exp\left\{-\frac{\varepsilon|\sigma|}{1+\varepsilon}\lambda_n\right\} = \int_0^{\infty} \exp\left\{-\frac{\varepsilon|\sigma|}{1+\varepsilon}t\right\} dn(t) \le \frac{\varepsilon|\sigma|}{1+\varepsilon}\int_0^{\infty} n(t)\exp\left\{-\frac{\varepsilon|\sigma|}{1+\varepsilon}t\right\}dt \le$$

$$\le \frac{\varepsilon|\sigma|}{1+\varepsilon}\int_{t_0}^{\infty} \exp\left\{-t\left(\frac{\varepsilon|\sigma|}{1+\varepsilon} - h\gamma(t)\right)\right\}dt + K_1(\varepsilon),$$

(3.40)

where $K_1(\varepsilon) = const > 0$. Since the function γ^{-1} is decreasing on $(0;\gamma(0))$, $\gamma^{-1}(x)\uparrow +\infty$ when $x \downarrow 0$, then

$$t(\sigma) = \gamma^{-1}\left(\frac{\varepsilon|\sigma|}{h(1+\varepsilon)^2}\right)\uparrow +\infty, \quad (\sigma \uparrow 0),$$

such that $t(\sigma) \ge t_0$, when $\sigma \ge \sigma_0(\varepsilon)$. Since $t\gamma(t)$ is non-decreasing, than

$$\int_{t_0}^{t(\sigma)} \exp\left\{-t\left(\frac{\varepsilon|\sigma|}{1+\varepsilon} - h\gamma(t)\right)\right\}dt \le \int_{t_0}^{t(\sigma)} \exp\{th\gamma(t)\}dt \le t(\sigma)\exp\{ht(\sigma)\gamma(t(\sigma))\} =$$

$$=\gamma^{-1}\left(\frac{\varepsilon|\sigma|}{h(1+\varepsilon)^2}\right)\exp\left\{\frac{\varepsilon|\sigma|}{h(1+\varepsilon)^2}\gamma^{-1}\left(\frac{\varepsilon|\sigma|}{h(1+\varepsilon)^2}\right)\right\}, \tag{3.41}$$

and

$$\int_{t(\sigma)}^{\infty}\exp\left\{-t\left(\frac{\varepsilon|\sigma|}{(1+\varepsilon)}-h\gamma(t)\right)\right\}dt=\int_{t(\sigma)}^{\infty}\exp\left\{-t\left(\frac{\varepsilon|\sigma|}{(1+\varepsilon)}-h\gamma(t(\sigma))\right)\right\}dt=$$

$$=\int_{t(\sigma)}^{\infty}\exp\left\{-\frac{\varepsilon^2|\sigma|}{(1+\varepsilon)^2}t\right\}dt=\frac{(1+\varepsilon)^2}{\varepsilon^2|\sigma|}\exp\left\{-\frac{\varepsilon^2|\sigma|t(\sigma)}{(1+\varepsilon)^2}\right\}\le\frac{(1+\varepsilon)^2}{\varepsilon^2|\sigma|}.$$

(3.42)

From the inequalities (3.40)-(3.41), when $\sigma\ge\sigma_0(\varepsilon)$, we obtain

$$\frac{M(\sigma)}{\mu\left(\frac{\sigma}{1+\varepsilon}\right)}\le=\frac{\varepsilon|\sigma|}{1+\varepsilon}\gamma^{-1}\left(\frac{\varepsilon|\sigma|}{h(1+\varepsilon)^2}\right)\exp\left\{\frac{\varepsilon|\sigma|}{h(1+\varepsilon)^2}\gamma^{-1}\left(\frac{\varepsilon|\sigma|}{h(1+\varepsilon)^2}\right)\right\}+\frac{1+\varepsilon}{\varepsilon}+K_1(\varepsilon),$$

whence, due to the fact that $x\gamma(x)\uparrow+\infty\ (x\to\infty)$, i. e. $|\sigma|\gamma^{-1}(|\sigma|)\uparrow+\infty\left(\sigma\uparrow 0\right)$, we get (3.39) with the constant $K(\varepsilon)\ge\frac{1+\varepsilon}{\varepsilon}+K_1(\varepsilon)$.

Theorem 3.11. *Let* $\sigma_a=0, \Phi\in\Omega(0)$ *and* $\ln\mu(\sigma,F)\le\Phi(\sigma)$ *for all* $\sigma<0$. *If*

$$\varlimsup_{n\to\infty}\frac{\ln n}{\lambda_n\Psi|(\varphi(\lambda_n))|}\le\beta_0<+\infty, \tag{3.43}$$

then for each $\varepsilon>0$ *there exists* $K(\varepsilon)$, *such that for all* $\sigma<0$

$$M(\sigma)\le\mu(\sigma)\exp\left\{\frac{\beta_0+\varepsilon}{1+\beta_0+\varepsilon}|\sigma|\Phi'\left(\Psi^{-1}\left(\frac{\sigma}{1+\beta_0+\varepsilon}\right)\right)\right\}+K(\varepsilon).$$

Proof. We have that $\ln|a_n|\le-\lambda_n\Psi(\varphi(\lambda_n)), n\ge 0$ and from (3.43) it follows that $\lambda_n\Psi(\varphi(\lambda_n))\ge\ln n/(\beta_0+\varepsilon/2)$, when $n\ge n_0(\varepsilon)$ and $\ln n(t)\le(\beta_0+\varepsilon)|\Psi(\varphi(t))|$, when $t\ge t_0(\varepsilon)$. Therefore, if we choose

$$\gamma(\sigma)=\Phi'\left(\Psi^{-1}\left(\frac{\sigma}{1+\beta_0+\varepsilon}\right)\right)\uparrow+\infty\quad\left(\sigma\uparrow 0\right),$$

then, when $\sigma \ge \sigma_0(\varepsilon)$, we will have

$$M(\sigma) \le \left(\sum_{\lambda_n < \gamma(\sigma)} + \sum_{\lambda_n \ge \gamma(\sigma)} \right) |a_n| \exp(\sigma \lambda_n) \le$$

$$\le \mu(\sigma) n(\gamma(\sigma)) + \sum_{\lambda_n \ge \gamma(\sigma)} \exp\left\{ -\lambda_n \Psi(\varphi(\lambda_n)) + \lambda_n \gamma^{-1}(\lambda_n) \right\} \le$$

$$\le \mu(\sigma) n(\gamma(\sigma)) + \sum_{n=n_0(\varepsilon)}^{\infty} \exp\left\{ -\lambda_n \Psi(\varphi(\lambda_n)) + (1+\beta_0+\varepsilon) \lambda_n \Psi(\varphi(\lambda_n)) \right\} =$$

$$= \mu(\sigma) n(\gamma(\sigma)) + \sum_{n=n_0(\varepsilon)}^{\infty} \exp\left\{ -\frac{\beta_0+\varepsilon}{\beta_0+\varepsilon/2} \ln n \right\} \le$$

$$\le \mu(\sigma) \exp\left\{ (\beta_0+\varepsilon) \Phi'\left(\Psi^{-1}\left(\frac{\sigma}{1+\beta_0+\varepsilon} \right) \right) \frac{|\sigma|}{1+\beta_0+\varepsilon} \right\} + K_1(\varepsilon),$$

therefore, we get (3.44) with some constant $K(\varepsilon) \ge K_1(\varepsilon)$.

Growth of entire Dirichlet series

The most used characteristics of the growth of Dirichlet series is the R-order

$$\rho_R = \overline{\lim_{\sigma \to +\infty}} \frac{1}{\sigma} \ln \ln M(\sigma, F),$$

and R-type (when $0 < \rho_R < +\infty$)

$$T_R = \overline{\lim_{\sigma \to +\infty}} \exp\{-\rho_R \sigma\} \ln M(\sigma, F).$$

Theorem 3.12. *If either* $\ln n = o(\lambda_n \ln \lambda_n)$, *or* $\ln n = o\left(\ln \frac{1}{|a_n|} \right)$, *when* $n \to \infty$, *then*

$$\rho_R = \overline{\lim_{\sigma \to +\infty}} \frac{\lambda_n \ln \lambda_n}{-\ln |a_n|}, \tag{3.45}$$

and if $\ln n = o(\lambda_n)$, *when* $n \to \infty$, *then*

$$T_R=\varlimsup_{\sigma\to+\infty}\frac{\lambda_n}{e\rho_R}|a_n|^{\rho_R/\lambda_n}. \tag{3.46}$$

Proof. Let $0<A,B<+\infty$ and $\Phi(\sigma)=Ae^{B\sigma}$. Then $\Phi\in\Omega(+\infty)$, $\varphi(x)=\frac{1}{B}\ln\frac{x}{AB}$, $\Psi(\sigma)=\sigma-\frac{1}{B}$, and $\Psi(\varphi(x))=\frac{1}{B}\ln\frac{x}{eAB}$. In order that $\ln\mu(\sigma,F)\le Ae^{B\sigma}$, when $\sigma\ge\sigma_0$, it is necessary and sufficient, that $\ln|a_n|\le-\frac{\lambda_n}{B}\ln\frac{\lambda_n}{eAB}$, when $n\ge n_0$. It follows that

$$\rho\overset{def}{=}\varlimsup_{\sigma\to+\infty}\frac{1}{\sigma}\ln\ln\mu(\sigma,F)=k\overset{def}{=}\varlimsup_{\sigma\to+\infty}\frac{\lambda_n\ln\lambda_n}{-\ln|a_n|} \tag{3.47}$$

and

$$T_R\overset{def}{=}\varlimsup_{\sigma\to+\infty}\exp\{-\rho_R\sigma\}\ln\mu(\sigma,F)=\kappa\overset{def}{=}T_R=\varlimsup_{\sigma\to+\infty}\frac{\lambda_n}{e\rho_R}|a_n|^{\rho_R/\lambda_n}. \tag{3.48}$$

For example, we'll prove (3.48). If $T<+\infty$, then $\ln\mu(\sigma,F)\le A\exp\{\rho_R\sigma\}$ for each $A>T$ and all $\sigma\ge\sigma_0(A)$. Therefore

$$\ln|a_n|\le-\frac{\lambda_n}{\rho_R}\ln\frac{\lambda_n}{eA\rho_R},\quad n\ge n_0 \tag{3.49}$$

that is, $\kappa\le A$, due to arbitrariness of A, we get the inequality $\kappa\le T$, which is obvious when $T=+\infty$. Further, if $\kappa<+\infty$, then for each $A>\kappa$ and for all $n\ge n_0(A)$ (3.49) is hold, therefore $\ln\mu(\sigma,F)\le A\exp\{\rho_R\sigma\}$ for all $\sigma\ge\sigma_0(A)$, that is, $T\le A$. Due to arbitrariness of A, we get the inequality $T\le\kappa$, which is obvious if $\kappa=+\infty$. So, $T=\kappa$.

The proof of the equality (3.47) is similar.

If $\ln n=o(\lambda_n)$, when $n\to\infty$, then by the Corollary 3.6 from the Theorem 3.9 and by the Cauchy's inequality we get $\mu(\sigma)\le M(\sigma)\le\mu(\sigma+\varepsilon)$ for each $\varepsilon>0$ and all $\sigma\ge\sigma_0(\varepsilon)$. Therefore

$$T \le T_R \le \overline{\lim_{\sigma\to+\infty}} \frac{\ln\mu(\sigma+\varepsilon)}{\exp\{\sigma\rho_R\}} = T\exp\{\varepsilon\rho_R\},$$

whence, taking into account the arbitrariness of ε, we have the equality $T = T_R$, and from (3.48) follows (3.46).

Suppose now, that $\rho < +\infty$. Then for each $B > \rho$ and all $\sigma \ge \sigma_0(B)$ the inequality $\ln\mu(\sigma) \le \Phi(\sigma) = \exp\{B_\sigma\}$ is hold and from the condition $\ln n = o(\lambda_n \ln\lambda_n)\,(n\to\infty)$ it follows that

$$\overline{\lim_{n\to+\infty}} \frac{n\ln n}{\lambda_n\Phi(\varphi(\lambda_n))} = \overline{\lim_{n\to+\infty}} \frac{B\ln n}{\lambda_n \ln(\lambda_n / eB)} = 0.$$

Therefore, by the Corollary 3.8 from the Theorem 3.9 $\mu(\sigma) \le M(\sigma) \le \mu\left(\frac{\sigma}{1-\varepsilon}\right)$ for each $\varepsilon > 0$ and all $\sigma \ge \sigma_0(\varepsilon)$. By the Corollary 3.7 of this theorem, the same inequality is hold, when $\ln n = o\left(\ln\frac{1}{|a_n|}\right)$, $n\to\infty$, and from it follows the equality $\rho_R = \rho$, which together with (3.47) gives (3.45). The theorem is proved.

For the entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ let $M_f(r) = \max\{|f(z)| : |z| = r\}$. The values

$$\rho = \overline{\lim_{\sigma\to+\infty}} \frac{\ln\ln M_f(r)}{\ln r}$$

and

$$T = \overline{\lim_{\sigma\to+\infty}} \frac{\ln M_f(r)}{r^\rho}, \quad 0 < \rho < \infty,$$

are called the order and type of the function f accordingly.

Corollary 3.10 (Hadamar's theorem). *The order and type of the entire function are calculated by the formulas*

$$\rho = \overline{\lim_{n\to+\infty}} \frac{n\ln n}{-\ln|a_n|},$$

and

$$T=\overline{\lim_{n\to+\infty}}\frac{n}{e\rho}|a_n|^{\rho/n}.$$

Let's retutn to the Dirichlet's series. The values

$$p_R=\overline{\lim_{\sigma\to+\infty}}\frac{\ln\ln M(\sigma,F)}{\ln\sigma},\quad q_R=\overline{\lim_{\sigma\to+\infty}}\sigma^{-p_R}\ln M(\sigma,F)$$

are called the logarithmic R-order and R-type respectively. It's easy to see that for each entire Dirichlet series $p_R\ge 1$. Therefore, in the definition of q_R, it is enough to require that $p_R<+\infty$.

Theorem 3.13. *If either* $\overline{\lim\limits_{n\to+\infty}}\dfrac{\ln\ln n}{\ln\lambda_n}\le 1$, or $\overline{\lim\limits_{n\to+\infty}}\dfrac{\ln n}{-\ln|a_n|}<1$, *then*

$$p_R=\overline{\lim_{\sigma\to+\infty}}\frac{\ln\lambda_n}{\ln\left(\frac{1}{\lambda_n}\ln\frac{1}{|a_n|}\right)}+1, \tag{3.50}$$

and if $1<p_R<+\infty$ *and either* $\ln n=o\left(\lambda_n^{p_R/(p_R-1)}\right)$, *or* $\ln n=o\left(\ln\dfrac{1}{|a_n|}\right)$, when $n\to\infty$, then

$$q_R=(p_R-1)^{p_R-1}p_R{}^{p_R}\overline{\lim_{\sigma\to+\infty}}\lambda_n^{p_R}\left(\ln\frac{1}{|a_n|}\right)^{1-p_R}. \tag{3.51}$$

Proof. Let’s choose the function $\Phi\in\Omega(+\infty)$, such that $\Phi(\sigma)=A\sigma^B$ for all sufficiently large σ, where A and B are positive constants and $B>1$. Then

$$\varphi(x)=\left(\frac{x}{AB}\right)^{1/(B-1)},\quad \Psi(\sigma)=\frac{B-1}{B}\sigma,$$

$$x\Phi(\varphi(x))=A(B-1)\left(\frac{x}{AB}\right)^{B/(B-1)}$$

for all sufficiently large *x*. In order that $\ln\mu(\sigma)\le A\sigma^B(\sigma\ge\sigma_0)$, it is necessary and sufficient that

$$\ln|a_n| \le -A(B-1)\left(\frac{\lambda_n}{AB}\right)^{B/(B-1)} \qquad (n \ge n_0).$$

Using the same method as when proving the Theorem 3.11, we obtain

$$p \overset{def}{=} \varlimsup_{\sigma\to+\infty} \frac{\ln\ln\mu(\sigma,F)}{\ln\sigma} = \varlimsup_{\sigma\to+\infty} \frac{\ln\lambda_n}{\ln\left(\frac{1}{\lambda_n}\ln\frac{1}{|a_n|}\right)} \tag{3.52}$$

and if $p>1$

$$q \overset{def}{=} \varlimsup_{\sigma\to+\infty} \sigma^{-p}\ln\mu(\sigma,F) = (p-1)^{p-1}p^p \varlimsup_{n\to+\infty} \lambda_n^p\left(\ln\frac{1}{|a_n|}\right)^{1-p}. \tag{3.53}$$

Suppose now that $p<+\infty$. Then $\ln\mu(\sigma)\le\Phi(\sigma)=\sigma^B$ for each $B>p$ and all sufficiently large σ. From the condition

$$\varlimsup_{n\to+\infty}(\ln\ln n/\ln\lambda_n)\le 1$$

it follows that $\ln n = o\left(\lambda_n^{1+\alpha}\right)$, $n\to\infty$, for each $\alpha>0$. Therefore

$$\varlimsup_{n\to+\infty}\frac{\ln n}{\lambda_n\Psi(\varphi(\lambda_n))} = \varlimsup_{n\to+\infty}\frac{1}{B-1}\left(\frac{B}{\lambda_n}\right)^{1+1/(B-1)}\ln n = 0,$$

and, using the Cauchy's inequality and the Corollary 3.8 from the Theorem 3.9, as when proving the Theorem 3.11, we get the equality $p_R = p$. From (3.52) it follows (3.50). If the condition

$$\varlimsup_{n\to+\infty}\left(\ln n/\ln\frac{1}{|a_n|}\right)<1$$

is hold, it is enough instead of the Collorary 3.8 to use the Corollary 3.7 from the Theorem 3.9.

The proof of equality $q = p_R$ is similar.

The most flexible growth scale of entire Dirichlet series is characterized by the generalized orders. Denote by L the class of continuous non-decreasing functions α, such that $\alpha(x)\ge 0$, when $x\ge x_0$, $\alpha(x)=\alpha(x_0)$, when $x\le x_0$ and the function α

increases to $+\infty$ on $[x_0;+\infty)$. We'll say that $\alpha \in L^0$, if $\alpha \in L$ and $\alpha\left(x\left(1+o(1)\right)\right)=\left(1+o(1)\right)\alpha(x)$, when $x \to +\infty$, further $\alpha \in L_{\text{ïç}}$, if $\alpha \in L$ and $\alpha(cx)=\left(1+o(1)\right)\alpha(x)$, when $x \to +\infty$; finally $\alpha \in L_{c\text{ïç}}$, if $\alpha \in L$ and $\alpha\left(x\alpha(x)\right)=\left(1+o(1)\right)\alpha(x)$, when $x \to +\infty$. It's easy to see that $L_{c\text{ïç}} \subset L_{\text{ïç}} \subset L^0$. Functions $L_{\text{ïç}}$ are called the slow-growing, and from $L_{c\text{ïç}}$ are called very slow-growing.

Let $\alpha \in L, \beta \in L$, G be any function defined on $[\sigma_0;+\infty)$. The value

$$\rho_{\alpha\beta}(G)=\overline{\lim_{\sigma\to+\infty}}\frac{\alpha\left(G(\sigma)\right)}{\beta(\sigma)}$$

is called the generalized order of the function G. If we take $G(\sigma)=\ln M(\sigma,F)$, then we obtain the definition of the generalized order $\rho_{\alpha\beta}(G)=\rho_{\alpha\beta}(\ln M)$ of entire Dirichlet series.

Denote

$$\kappa_{\alpha\beta}(F)=\overline{\lim_{n\to+\infty}}\frac{\alpha(\lambda_n)}{\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|a_n|}\right)}.$$

Imposing certain restrictions on the functions α and β, we can find the conditions for λ_n or a_n, under which $\rho_{\alpha\beta}(F)=\kappa_{\alpha\beta}(F)$.

Theorem 3.14. *Let $\alpha \in L^0$ and $\beta \in L^0$ such that for each $\rho \in (0;+\infty)$*

$$\overline{\lim_{\sigma\to+\infty}}\frac{1}{\beta(\sigma)}\alpha\left(\int_{\sigma_0}^{\sigma}\alpha^{-1}\left(\rho\beta(t)\right)dt\right)\le\rho\ , \tag{3.54}$$

where σ_0 is the number larger, than $\beta^{-1}\left(\frac{1}{\rho}\alpha(x_0)\right)$. If either $\ln n=o\left(\ln\frac{1}{|a_n|}\right), n\to\infty$, or for each $\rho\in(0;+\infty)$ and some $t_0\ge\alpha^{-1}\left(\rho\beta(x_0)\right)$

$$\ln n = o\left(\int\limits_{t_0}^{\lambda_n} \beta^{-1}\left(\frac{1}{\rho}\alpha(t)\right)dt\right), \quad n\to\infty \ , \tag{3.55}$$

then $\rho_{\alpha\beta}(F) = k_{\alpha\beta}(F)$.

Proof. Let $\rho_{\alpha\beta}(F) < +\infty$. Then $\ln M(\sigma, F) \le \alpha^{-1}(\rho\beta(\sigma))$ for each $\rho > \rho_{\alpha\beta}(F)$ and all $\sigma \ge \sigma_0(\rho)$. Therefore, by the Cauchy's inequality $\ln|a_n| \le \alpha^{-1}(\rho\beta(\sigma)) - \sigma\lambda_n$ for all $n \ge 0$ and $\sigma \ge \sigma_0(\rho)$. When $\sigma = \beta^{-1}\left(\frac{1}{\rho}\alpha(\lambda_n)\right), n \ge n_0(\rho)$, hence

$$\ln|a_n| \le \lambda_n - \lambda_n\beta^{-1}\left(\frac{1}{\rho}\alpha(\lambda_n)\right),$$

i.e.

$$\alpha(\lambda_n) \le \beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|a_n|} + 1\right).$$

Since $\beta \in L^0$ and ρ is an arbitrary number, then from the last inequality we get the inequality $k_{\alpha\beta}(F) \le \rho_{\alpha\beta}(F)$, which is obvious if $\rho_{\alpha\beta}(F) = +\infty$.

Suppose $k_{\alpha\beta}(F) \ne \rho_{\alpha\beta}(F)$. Then $k_{\alpha\beta}(F) < \rho_{\alpha\beta}(F)$. If we take $k_{\alpha\beta}(F) < \rho < \rho_{\alpha\beta}(F)$, then given the growth of the function $\beta^{-1}\left(\frac{1}{\rho}\alpha(x)\right)$ for $n \ge n_0(\rho)$, we'll get

$$\ln|a_n| \le -\lambda_n\beta^{-1}\left(\frac{1}{\rho}\alpha(\lambda_n)\right) \le \int\limits_{t_0}^{\lambda_n} \beta^{-1}\left(\frac{1}{\rho}\alpha(t)\right)dt + const \ . \tag{3.56}$$

Note that if $\Phi \in \Omega(A)$ and for $\sigma \in [\sigma_0; A)$

$$\Phi(\sigma) = \int\limits_{\sigma_0}^{\sigma} \omega(t)dt + const, \tag{3.57}$$

where ω is positive, continuous and increasing to $+\infty$ function on $[\sigma_0;A)$, then $\varphi(x)=w^{-1}(x)$, when $x\ge x_0$, and since $\left(x\Psi(\varphi(x))\right)'=\left(x\varphi(x)-\Phi(\varphi(x))\right)'=\varphi(x)$, then

$$x\Psi(\varphi(x))=\int\limits_{x_0}^{x}\omega^{-1}(t)dt+const\,. \tag{3.58}$$

In order that for all $\sigma\in[\sigma_0;A)$

$$\ln\mu(\sigma,F)=\int\limits_{\sigma_0}^{\sigma}\omega(t)dt+const\,,$$

it necessary and sufficient, that

$$\ln|a_n|\le-\int\limits_{x_0}^{\lambda_n}\omega^{-1}(t)dt+const\,.$$

Using this statement, from (3.56) we get the inequality

$$\ln\mu(\sigma,F)\le\int\limits_{\sigma_0}^{\sigma}\alpha^{-1}(\rho\beta(t))dt+const\,,$$

whence, due to the conditions $\alpha\in L^0$ and (3.54), we get the inequality $\rho_{\alpha\beta}(\ln\mu)\le\rho$. Finally, given (3.36) with $\omega(x)=\beta^{-1}\left(\frac{1}{\rho}\alpha(x)\right)$, the condition (3.55) is equivalent to the condition (3.36) $\beta_0=0$. By the Collorary 3.8 from the Theorem 3.9

$$\ln M(\sigma,F)\le\ln\mu\left((1+o(1))\sigma,F\right),\quad\sigma\to+\infty\,. \tag{3.59}$$

The relation (3.59) by the Collorary 3.7 from the Theorem 3.9 is hold also if $\ln n=o\left(\ln\frac{1}{|a_n|}\right), n\to\infty$. From it, by the condition $\beta\in L^0$, the inequality $\rho_{\alpha\beta}(F)\le\rho_{\alpha\beta}(\ln\mu)\le\rho$ follows, which is impossible. The equality $\rho_{\alpha\beta}(F)=k_{\alpha\beta}(F)$ is proved.

Remark 1. Since

$$\int_{\sigma_0}^{\sigma} \alpha^{-1}\left(\rho\beta(t)\right)dt \le \sigma\alpha^{-1}\left(\rho\beta(\sigma)\right) + const,$$

then the condition (3.54) takes place, if $\alpha \in L_{cï\,ç}$.

Rematk 2. If $\alpha \in L_{ï\,ç}$ and $\dfrac{d\beta^{-1}\left(\frac{1}{\rho}\alpha(x)\right)}{d\ln x} = O(1)$, when $x \to +\infty$ for each $\rho \in (0;+\infty)$, then (3.54) is hold and (3.55) can be replaced by a condition

$$\ln n = o\left(\lambda_n \beta^{-1}\left(\frac{1}{\rho}\alpha(\lambda_n)\right)\right), \quad n \to \infty. \tag{3.60}$$

Indeed, since $\alpha \in L_{ï\,ç}$, then $\dfrac{\alpha^{-1}(\rho x)}{\alpha^{-1}((\rho+\varepsilon)x)} \to 0$, when $x \to +\infty, \varepsilon > 0$, because if $\dfrac{\alpha^{-1}(\rho x_k)}{\alpha^{-1}((\rho+\varepsilon)x_k)} \ge h > 0$ for some increasing to $+\infty$ sequence (x_k), then we would have $\rho x_k \ge \alpha\left(h\alpha^{-1}((\rho+\varepsilon)x_k)\right) = (1+o(1))(\rho+\varepsilon)x_k, k \to \infty$, that is impossible.

Further, the condition $\dfrac{d\beta^{-1}\left(\frac{1}{\rho}\alpha(x)\right)}{d\ln x} = O(1)$, when $x \to +\infty$ is an equivalent to the condition $\dfrac{d\ln\alpha^{-1}(\rho\beta(x))}{dx} \ge c > 0, x \ge x_0$, and therefore for each $\varepsilon > 0$ we have

$$\varlimsup_{\sigma\to+\infty} \frac{\int_{\sigma_0}^{\sigma} \alpha^{-1}(\rho\beta(t))dt}{\alpha^{-1}((\rho+\varepsilon)\beta(\sigma))} \le \varlimsup_{\sigma\to+\infty} \frac{\alpha^{-1}(\rho\beta(\sigma))}{\left(\alpha^{-1}((\rho+\varepsilon)\beta(\sigma))\right)'} \le$$

$$\le \varlimsup_{\sigma\to+\infty} \frac{\alpha^{-1}((\rho+\varepsilon)\beta(\sigma))}{\left(\alpha^{-1}((\rho+\varepsilon)\beta(\sigma))\right)'} \varlimsup_{\sigma\to+\infty} \frac{\alpha^{-1}(\rho\beta(\sigma))}{\alpha^{-1}((\rho+\varepsilon)\beta(\sigma))} \le$$

$$\le \frac{1}{c} \varlimsup_{\sigma\to+\infty} \frac{\alpha^{-1}(\rho(x))}{\left(\alpha^{-1}((\rho+\varepsilon)x)\right)} = 0.$$

It follows that

$$\int_{\sigma_0}^{\sigma} \alpha^{-1}\left(\rho\beta(t)\right)dt \le \alpha^{-1}\left((\rho+\varepsilon)\beta(\sigma)\right)$$

for each $\varepsilon > 0$and all $\sigma \ge \sigma_0(\varepsilon)$. Therefore, the condition (3. 54) is hold.

Further,

$$\varlimsup_{x\to+\infty} \frac{\int_{x_0}^{x} \beta^{-1}\left(\frac{1}{\rho}\alpha(t)\right)dt}{x\beta^{-1}\left(\frac{1}{\rho}\alpha(x)\right)} = \varlimsup_{x\to+\infty} \frac{\beta^{-1}\left(\frac{1}{\rho}\alpha(x)\right)}{\beta^{-1}\left(\frac{1}{\rho}\alpha(x)\right)+x\left(\beta^{-1}\left(\frac{1}{\rho}\alpha(x)\right)\right)'} = 1,$$

whence the equivalence of the conditions (3.60) and (3.55) follows.

Growth of Dirichlet series, absolutely convergent in the half-plane

Let the Dirichlet series (3.1) now have a zero abscissa of absolute convergence. By R-order and R-type of such Dirichlet series will'll call

$$\rho_R^0 = \varlimsup_{\sigma\to 0} |\sigma| \ln\ln M(\sigma, F)$$

and

$$T_R^0 = \varlimsup_{\sigma\to 0} \exp\left\{-\frac{\rho_R^0}{|\sigma|}\right\} \ln M(\sigma, F), \quad 0 < \rho_R^0 < +\infty,$$

and the logarithmic R-order and R-type we'll call

$$p_R^0 = \varlimsup_{\sigma\to 0} \frac{\ln\ln M(\sigma, F)}{\ln(1/|\sigma|)}$$

and

$$q_R^0 = \varlimsup_{\sigma\to 0} |\sigma|^{p_R^0} \ln M(\sigma, F), \quad 0 < p_R^0 < +\infty.$$

The logarithmic R-order and R-type are often called simply as order and type.

Theorem 3.15. *If either* $\underline{\lim}_{n\to+\infty}\left(\ln|a_n|/\ln n\right)>0$*, or* $\ln\ln n=o\left(\ln\lambda_n\right), n\to\infty$*, then*

$$p_R^0=\overline{\lim_{\sigma\to+\infty}}\frac{\ln\lambda_n}{\ln\left(\frac{\lambda_n}{\ln|a_n|}\right)}-1, \tag{3.61}$$

and if either $\ln n=o\left(\lambda_n^{p_R^0/\left(p_R^0+1\right)}\right), n\to\infty r$ *, or* $\underline{\lim}_{n\to\infty}\left(\ln|a_n|/\ln n\right)=+\infty$ *, then*

$$q_R^0=\left(p_R^0+1\right)^{-\left(p_R^0-1\right)}\left(p_R^0\right)^{p_R^0}\overline{\lim_{n\to+\infty}}\lambda_n^{-p_R^0}\left(\ln|a_n|\right)^{1+p_R^0}. \tag{3.62}$$

Proof. Choose $\Phi(\sigma)=A|\sigma|^{-B}$, where A and B are positive constants. Then $\Phi\in\Omega(0)$,

$$\varphi(x)=-\left(\frac{AB}{x}\right)^{1/(B+1)}, \Psi(\sigma)=\frac{B+1}{B}\sigma,$$

$$x\Psi\left(\varphi(x)\right)=-\frac{B+1}{B}x\left(\frac{AB}{x}\right)^{1/(B+1)}=-\left(B+1\right)A^{1/(B+1)}\left(\frac{x}{B}\right)^{B/(B+1)}.$$

In order that $\ln\mu(\sigma)\le A|\sigma|^{-B}, \left(0>\sigma>\sigma_0\right)$, it is necessary and sufficient that

$$\ln|a_n|\le\frac{B+1}{B}\lambda_n\left(\frac{AB}{\lambda_n}\right)^{1/(B+1)}\qquad\left(n\ge n_0\right).$$

Using the same method as when proving the Theorem 3.12, we have

$$p^0\overset{def}{=}\overline{\lim_{\sigma\to0}}\frac{\ln\ln\mu(\sigma,F)}{\ln\left(1/|\sigma|\right)}=\overline{\lim_{n\to\infty}}\frac{\ln\lambda_n}{\ln\left(\frac{\lambda_n}{\ln|a_n|}\right)}-1, \tag{3.63}$$

and

$$q\overset{def}{=}\overline{\lim_{\sigma\to0}}|\sigma|^{p^0}\ln\mu(\sigma,F)=\left(p^0+1\right)^{-\left(p^0+1\right)}\left(p^0\right)^{p^0}\overline{\lim_{n\to+\infty}}\lambda_n^{-p^0}\left(\ln|a_n|\right)^{p^0+1}. \tag{3.64}$$

If $\underline{\lim}_{n\to\infty}\left(\ln|a_n|/\ln n\right)>0$, then (3.36) is hold with $h_0<+\infty$ and by the Collorary 3.9 from the Theorem 3.9 we have (3.37), whence given the Cauchy's inequality easily follows $p^0=p_R^0$. If $\underline{\lim}_{n\to\infty}\left(\ln|a_n|/\ln n\right)=+\infty$, then from the same corollary follows the equality $q^0=q_R^0$.

Further, if $\ln\ln n=o(\ln\lambda_n),\, n\to\infty$, then

$$\overline{\lim_{n\to\infty}}\frac{\ln n}{\lambda_n\left|\Psi(\varphi(\lambda_n))\right|}=\frac{B}{B+1}(AB)^{-1/(B+1)}\overline{\lim_{n\to\infty}}\lambda_n^{-B/(B+1)}\ln n=0,$$

$B>0$. The same inequality is true, if $B=p_R^0$ and $\ln n=o\left(\lambda_n^{p_R^0/(p_R^0+1)}\right), n\to\infty$.

Therefore, by the Theorem 2.4 for each $\varepsilon>0$ we receive

$$\ln M(\sigma)\le\ln\mu(\sigma)+\frac{\varepsilon|\sigma|}{1+\varepsilon}\Phi'\left(\Psi^{-1}\left(\frac{\sigma}{1+\varepsilon}\right)\right)+K_1(\varepsilon)=$$

$$=\ln\mu(\sigma)+\varepsilon A(B+1)^{B+1}B^{-B}(1+\varepsilon)^{B}|\sigma|^{-B}+K_1(\varepsilon), \tag{3.65}$$

where $K_1(\varepsilon)$ is the positive number. It follows that

$$\ln\ln M(\sigma)\le\max\left\{\ln\ln\mu(\sigma),B^*\ln\frac{1}{|\sigma|}\right\}+O(1),\quad \sigma\to 0,$$

where B^* is arbitrary number, $B^*>p^0$. From the last relations and Cauchy's inequality, due to the arbitrariness of B^*, it easily follows that $p_R^0=p^0$, and therefore from (3.63) we get (3.61). If we take $B=p_R^0=p^0$ and $A>q^0$, then from (3.65) we'll get $q_R^0\le q^0+\varepsilon A$, that is, due to the arbitrariness of ε and the inequality $q^0\le q_R^0$, the equality $q_R^0=q^0$ takes place. Therefore, from (3.64) we get (3.62). Theorem 3.15 is proved.

For the analytic in the circle $\{z:|z|<1\}$ function $f(z)=\sum_{n=0}^{\infty}a_nz^n$ the values

$$p=\overline{\lim_{\sigma\to+\infty}}\frac{\ln\ln M_f(r)}{-\ln(1-r)}$$

and

$$q=\overline{\lim_{\sigma\to+\infty}}(1-r)^{\rho}\ln M_f(r),\quad 0<\rho<\infty$$

are called the order and type accordingly. From the Theorem 2.8 such a corollary easily follows.

Corollary 3.11. *The order and type of analytic in the circle function are calculated by the formulas*

$$p=\overline{\lim_{n\to+\infty}}\frac{\ln n}{\ln\left(\dfrac{n}{\ln|a_n|}\right)}-1$$

and

$$q=(p+1)^{-(p+1)}(p)^{p}\overline{\lim_{n\to+\infty}}n^{-p}\left(\ln|a_n|\right)^{1+p}.$$

Theorem 3.16. *If either* $\ln n=o(\lambda_n/\ln\lambda_n),\, n\to\infty$, *or* $\ln n=o(\ln|a_n|),\, n\to\infty$

, then

$$\rho_R^0=\overline{\lim_{n\to\infty}}\frac{\ln|a_n|}{\lambda_n}\ln\lambda_n.$$

Theorem 3.17. *If* $\ln n(t)=O(\ln t),\, t\to+\infty$, *then*

$$T_R^0=\rho_R^0\exp\left\{\kappa^0-1\right\},\quad \kappa^0=\overline{\lim_{n\to\infty}}\left(\frac{\ln|a_n|}{\rho_R^0\lambda_n}\ln\frac{\lambda_n}{(\ln\lambda_n)^2}-1\right)\lambda_n.$$

At the end of this chapter, we'll give two theorems about the generalized orders of completely converging in the half-plane Dirichlet series.

Let $\alpha\in L,\beta\in L$. The value

$$\rho_{\alpha\beta}^0(F)=\overline{\lim_{\sigma\uparrow 0}}\frac{\alpha(\ln M(\sigma,F))}{\beta(1/|\sigma|)}\tag{3.66}$$

is called the generalized order of the Dirichlet series absolutely convergent in the half-plane.

Theorema 3.18. *Let the functions* $\alpha \in L$ *and* $\beta \in L$ *satisfy the conditions*

$$\frac{x}{\beta^{-1}(\rho\alpha(x))} \uparrow +\infty, \quad (x_0(\rho) \le x \to +\infty) \tag{3.67}$$

and

$$\alpha\left(\frac{x}{\beta^{-1}(\rho\alpha(x))}\right) = (1+o(1))\alpha(x) \quad (x \to +\infty) \tag{3.68}$$

for each $\rho \in (0;+\infty)$. *Let the Dirichlet series (3.1) has the abscissa* $\sigma_a = 0$ *of absolute convergence. If either* $\underline{\lim}_{n\to\infty} (\ln|a_n| / \ln n) > 0$, *or*

$$\alpha(\lambda_n) = o\left(\beta\left(\frac{\lambda_n}{\ln n}\right)\right), \quad n \to \infty,$$

then

$$\rho^0_{\alpha\beta}(F) = k^{(1)}_{\alpha\beta}(F) \overset{def}{=} \overline{\lim_{n\to+\infty}} \frac{\alpha(\lambda_n)}{\beta\left(\dfrac{\lambda_n}{\ln|a_n|}\right)}.$$

The conditions (3.67) and (3.68) satisfy, for example, the functions $\alpha(x) = \ln\ln x$ and $\beta(x) = \ln x$.

Theorem 3.19. *Let the functions* $\alpha \in L$ *and* $\beta \in L$ *for each* $\rho \in (0;+\infty)$ *satisfy the conditions*

$$\frac{x}{\alpha^{-1}(\rho\beta(x))} \uparrow +\infty, \quad (x_0(\rho) \le x \to +\infty) \tag{3.69}$$

and

$$\beta\left(\frac{x}{\alpha^{-1}(\rho\beta(x))}\right) = (1+o(1))\beta(x) \quad (x \to +\infty), \tag{3.70}$$

and the Dirichlet series (3.1) has an abscissa $\sigma_a = 0$ *of absolute convergence. If either* $\underline{\lim}_{n\to\infty} (\ln|a_n| / \ln n) > 0$, *or* $\alpha(\ln n) = o(\beta(\lambda_n)), n \to \infty$, *then*

$$\rho^0_{\alpha\beta}(F) = k^{(2)}_{\alpha\beta}(F) \overset{def}{=} \varlimsup_{n\to+\infty} \frac{\alpha(\ln|a_n|)}{\beta(\lambda_n)}.$$

Conditions (3.69) and (3.70) satisfy, for example, the functions $\alpha(x) = \ln x$ and $\beta(x) = \ln\ln x$.

Convergence classes

We will get a connection between the growth of the maximum modulus and the behavior of the coefficients in terms of convergence classes.

Hardy's inequality

Let $p > 1$, $q = p/(p-1)$ and f be the positive function on $(A;B)$, $-\infty \le A < B \le +\infty$. Let (λ^*_n) be the sequence of the positive numbers, (a_n) be the sequence of the numbers from $(A;B)$. Then

$$A_n \overset{def}{=} \frac{\lambda^*_1 a_1 + \cdots + \lambda^*_n a_n}{\lambda^*_1 + \cdots + \lambda^*_n} \in (A;B).$$

The well-known Hardy’s inequality we’ll get from such a theorem.

Theorem 3.20. *If the function* $f^{1/p}$ *is convex on* $(A;B)$*, and the sequence* (μ_n) *is positive and non-increasing, then for each* $\omega \le +\infty$

$$\sum_{n=1}^{\omega} \mu_n \lambda^*_n f(A_n) \le q^p \sum_{n=1}^{\omega} \mu_n \lambda^*_n f(a_n) . \qquad (3.71)$$

Proof. Denote $t = \lambda^*_1 + \cdots + \lambda^*_n$, $n \ge 1$. Then

$$A_n = \frac{\lambda^*_1 a_1 + \cdots + \lambda^*_{n-1} a_{n-1}}{t_n} + \frac{\lambda^*_n a_n}{t_n} = \frac{t_{n-1}}{t_n} A_{n-1} + \frac{\lambda^*_n}{t_n} a_n .$$

Since $\frac{t_{n-1}}{t_n}+\frac{\lambda_n^*}{t_n}=1$, and the function $f^{1/p}$ is convex, then

$$f^{1/p}(A_n)\le\frac{t_{n-1}}{t_n}f^{1/p}(A_{n-1})+\frac{\lambda_n^*}{t_n}f^{1/p}(a_n),$$

whence

$$-f^{1/p}(a_n)\le\frac{t_{n-1}}{\lambda_n^*}f^{1/p}(A_{n-1})-\frac{t_n}{\lambda_n^*}f^{1/p}(A_n).$$

So, taking into account the equality $1/p+1/q=1$, we have

$$Q_n\overset{def}{=}\lambda_n^*f(A_n)-q\lambda_n^*f^{1/p}(a_n)f^{1/p}(A_n)\le$$

$$\le\lambda_n^*f(A_n)+q\lambda_n^*\left(\frac{t_{n-1}}{\lambda_n^*}f^{1/q}(A_n)f^{1/p}(A_{n-1})-\frac{t_n}{\lambda_n^*}f^{1/q}(A_n)f^{1/p}(A_n)\right)=$$

$$=\lambda_n^*f(A_n)-qt_nf(A_n)+qt_{n-1}f^{1/q}(A_n)f^{1/p}(A_{n-1})=$$

$$=\left(\lambda_n^*-qt_n\right)f(A_n)+qt_{n-1}f^{1/q}(A_n)f^{1/p}(A_{n-1}).\qquad(3.72)$$

Let's consider the function $u(x)=\frac{1}{p}x^p-ax+\frac{1}{q}a^q$ on $[0;+\infty)$, where a is any non negative number. This function has a unique minimal point $x=x(a)=a^{1/(p-1)}$, and $u(x(a))=0$. It follows that for all $a\ge0$ and $x\ge0$ the inequality $\frac{1}{p}x^p+\frac{1}{q}a^q\ge ax$ is true. Therefore, from (3.2) we have $Q_1=-(q-1)\lambda_1^*f(A_1)$ and

$$Q_n\le\left(\lambda_n^*-qt_n\right)f(A_n)+qt_{n-1}\left(\frac{1}{p}f(A_{n-1})+\frac{1}{q}f(A_n)\right)=$$

$$=\left(\lambda_n^*+t_{n-1}-qt_n\right)f(A_n)+\frac{q}{p}t_{n-1}f(A_{n-1})=$$

$$=(1-q)t_nf(A_n)+\frac{q}{p}t_{n-1}f(A_{n-1})=\frac{1}{p-1}\left(t_{n-1}f(A_{n-1})-t_nf(A_n)\right),\quad n\ge2,$$

and since the sequence μ_n is non-increasing, then when $N<+\infty$

$$\sum_{n=1}^{N}\mu_nQ_n \le \frac{1}{p-1}\sum_{n=2}^{N}\mu_n\left(t_{n-1}f\left(A_{n-1}\right)-t_nf\left(A_n\right)\right)+\mu_1Q_1 =$$

$$=\frac{1}{p-1}\sum_{n=2}^{N-1}\left(\mu_n-\mu_{n-1}\right)t_nf\left(A_n\right)-\mu_Nt_Nf\left(A_N\right)<0$$

is hold.

By the definition Q_n and the Holder's inequality

$$\sum_{n=1}^{N}a_nb_n \le \left(\sum_{n=1}^{N}a_n^p\right)^{1/p}\left(\sum_{n=1}^{N}b_n^q\right)^{1/q},$$

we get

$$\sum_{n=1}^{N}\mu_n\lambda_n^*f\left(A_n\right)\le q\sum_{n=1}^{N}\mu_n\lambda_n^*f^{1/p}\left(a_n\right)f^{1/q}\left(A_n\right)=$$

$$=q\sum_{n=1}^{N}\left(\mu_n\lambda_n^*f\left(a_n\right)\right)^{1/p}\left(\mu_n\lambda_n^*f\left(A_n\right)\right)^{1/q}\le q\left(\sum_{n=1}^{N}\mu_n\lambda_n^*f\left(a_n\right)\right)^{1/p}\left(\sum_{n=1}^{N}\mu_n\lambda_n^*f\left(A_n\right)\right)^{1/q}.$$

If we divide the inequality by the last multiplier and raise it to the power p, we'll get the inequality (3.71) for $\omega = N$, and due to the arbitrariness of N, the inequality (3.71) is proved for all ω .

Corollary 3.12. *Let* $p>1, a_n \ge 0$ *and* $A_n = \frac{1}{n}\left(a_1+\cdots a_n\right)>0$. *Then*

$$\sum_{n=1}^{\omega}A_n^p \le \left(\frac{p}{p-1}\right)^p\sum_{n=1}^{\omega}a_n^p \quad . \tag{3.73}$$

The inequality (3.73) is called the Hardy's equality and follows from (3.71), if $\lambda_n^*=1,\ \mu_n=1\left(n\ge 1\right)$ and $f\left(x\right)=x^p$.

Dirichlet series of finite R-order or finite logarithmic R-order

Let the entire function f, represented by the series $f(z)=\sum_{n=0}^{\infty} a_n z^n$, $z=re^{i\varphi}$, have the finite order $\rho>0$ and $M_f(r)=\max\{|f(z)|:|z|=r\}$. The function f belongs to the convergence class, if

$$\int_1^{\infty} \frac{\ln M_f(r)}{r^{\rho+1}} dr < +\infty . \tag{3.74}$$

In 1923 G. Valiron proved that if f belongs to the convergence class, then

$$\sum_{n=1}^{\infty} |a_n|^{\rho/n} < +\infty . \tag{3.75}$$

The analogue of the ratio (3.74) for the entire Dirichlet series of the finite R-order $\rho_R>0$ is the ratio

$$\int_0^{\infty} \frac{\ln M(\sigma,F)}{\exp(\rho_R \sigma)} d\sigma < +\infty \ . \tag{3.76}$$

In order to find the conditions for the coefficients of Dirichlet series, at which the ratio (3.76) and its analogues for the Dirichlet series, completely convergent in the half-plane $\{s:\operatorname{Re} s<0\}$, both the finite R-order and the finite logarithmic R-order, are hold, we first prove such a theorem.

Theorem 3.21. *Let the series (3.1) have the abscissa* $\sigma_a=\alpha_0=A\in(-\infty;+\infty]$ *of absolute convergence, where* α_0 *is defined by the equality (3.14). Let* $\mu(\sigma,F)$ *be its maximum term, and* a_n^0 *are the coefficients of Newton's majorant. Let* β *be the positive continuous non-decreasing function on* $[a,A)$, *such that integrals*

$$\int_a^A \frac{d\sigma}{\beta(\sigma)}, \quad \int_a^A \frac{\sigma d\sigma}{\beta(\sigma)} \tag{3.77}$$

are convergent. Then, in order that

$$\int_a^A \frac{\ln \mu(\sigma)}{\beta(\sigma)} d\sigma < +\infty , \tag{3.78}$$

it is necessary and sufficient that

$$\sum_{n=n_0}^{\infty}(\lambda_n-\lambda_{n-1})B\left(\frac{1}{\lambda_n}\ln\frac{1}{a_n^0}\right)<+\infty, \tag{3.79}$$

where

$$B(x)=\int_x^A\frac{\sigma-x}{\beta(\sigma)}d\sigma.$$

Proof. The Dirichlet series (3.1) and its Newton's majorant (3.11) have the same maximum terms $\nu(\sigma)=\nu(\sigma,F)=\nu(\sigma,F_{\text{ì í}})$ and the central indexes $\mu(\sigma)=\mu(\sigma,F)=\mu(\sigma,F_{\text{ì í}})$ and $|a_n|\le a_n^0$ for all n.

Suppose first that the Newton's diagram of the series (3.1) consists of the infinite number of edges. From the construction of the Newton's diagram, it is clear that the angular coefficients of these edges are

$$\kappa_n^0=\kappa_n^0(F_{\text{ì í}})=\frac{\ln a_n^0-\ln a_{n+1}^0}{\lambda_{n+1}-\lambda_n}=\frac{L(\lambda_{n+1})-L(\lambda_n)}{\lambda_{n+1}-\lambda_n}.$$

Since $\sigma_a=\alpha_0=A$, the sequence $\left(\kappa_n^0\right)$ increases to A. Without lost the generality, we can assume that $a_0=a_0^0=1$. Then $\kappa_0^0=\frac{1}{\lambda_1}\ln\frac{1}{a_1^0}$ and again without lost the generality, we can assume that $\kappa_0^0\ge a$, because otherwise we can properly presesce the function β. Using the equality (3.10), we have

$$\int_a^A\frac{\ln\mu(\sigma)}{\beta(\sigma)}d\sigma=\int_a^A\frac{1}{\beta(\sigma)}\left(\int_a^\sigma\lambda_{\nu(t)}dt+\ln\mu(a)\right)d\sigma=$$

$$=\int_a^A\lambda_{\nu(t)}\int_t^A\frac{d\sigma}{\beta(\sigma)}dt+const=\int_a^A\lambda_{\nu(t)}\beta_1(t)dt+const,$$

where

$$\beta_1(t)=\int_t^A\frac{d\sigma}{\beta(\sigma)}$$

is decreasing to 0 function. By the Theorem 3.8 it follows, that

$$\int_a^A \frac{\ln \mu(\sigma)}{\beta(\sigma)} d\sigma = \sum_{n=1}^{\infty} \lambda_n \int_{\kappa_{n-1}^0}^{\kappa_n^0} \beta_1(t)\, dt + const =$$

$$= \sum_{n=1}^{\infty} \lambda_n \left(B\left(\kappa_{n-1}^0\right) - B\left(\kappa_n^0\right)\right) + const = \sum_{n=1}^{\infty} (\lambda_n - \lambda_{n-1}) B\left(\kappa_{n-1}^0\right) + const, \qquad (3.80)$$

where

$$B(x) = \int_x^A \beta_1(t)\, dt = \int_x^A \int_t^A \frac{d\sigma}{\beta(\sigma)} dt = \int_x^A \frac{d\sigma}{\beta(\sigma)} \int_x^{\sigma} dt = \int_x^A \frac{\sigma - x}{\beta(\sigma)} d\sigma .$$

Denote $B_1(x) = \sqrt{B(x)}$. Then $B_1'(x) = \frac{1}{2} B(x)^{-1/2} B'(x)$ and

$$B_1''(x) = -\frac{1}{4} B^{-3/2}(x) \left(B'(x)\right)^2 + \frac{1}{2} B^{-1/2}(x) B''(x) =$$

$$= \frac{1}{2} B^{-3/2}(x) \left(B(x) B''(x) - \frac{1}{2} \left(B'(x)\right)^2 \right) =$$

$$= \frac{1}{2} B^{-3/2}(x) \left(-\int_x^A \beta_1(t)\, dt \beta_1'(x) - \frac{1}{2} \beta_1^2(x) \right) \geq$$

$$\geq \frac{1}{2} B^{-3/2}(x) \left(-\int_x^A \beta_1(t) \beta_1'(t)\, dt - \frac{1}{2} \beta_1^2(x) \right) = 0 ,$$

because $\beta_1'(x) = -1/\beta(x)$ is the decreasing function. So, the function $B^{1/2}$ is convex on $[a; A)$.

Further

$$\ln a_n^0 = \ln a_n^0 - \ln a_{n-1}^0 + \cdots + \ln a_1^0 - \ln a_0^0 + \ln a_0^0 =$$

$$= -\kappa_{n-1}^0 (\lambda_n - \lambda_{n-1}) - \cdots - \kappa_0^0 (\lambda_1 - \lambda_0),$$

i.e.

$$\frac{1}{\lambda_n} \ln \frac{1}{a_n^0} = \frac{\kappa_0^0 \lambda_1^* + \cdots + \kappa_{n-1}^0 \lambda_n^*}{\lambda_1^* + \cdots + \lambda_n^*}, \quad \lambda_n^* = \lambda_n - \lambda_{n-1} . \qquad (3.81)$$

Therefore, by the Theorem 3.1, $f(x)=B(x)$, $p=2$ and $\mu_n=1, n\ge 1$ we obtain

$$\sum_{n=1}^{\omega}\left(\lambda_n-\lambda_{n-1}\right)B\left(\frac{1}{\lambda_n}\ln\frac{1}{a_n^0}\right)\le 4\sum_{n=1}^{\omega}\left(\lambda_n-\lambda_{n-1}\right)B\left(\kappa_{n-1}^0\right).$$

On the other hand $\frac{1}{\lambda_n}\ln\frac{1}{a_n^0}\le\kappa_{n-1}^0$ and since the function B is descreasing, then

$$\sum_{n=1}^{\omega}\left(\lambda_n-\lambda_{n-1}\right)B\left(\frac{1}{\lambda_n}\ln\frac{1}{a_n^0}\right)\ge\sum_{n=1}^{\omega}\left(\lambda_n-\lambda_{n-1}\right)B\left(\kappa_{n-1}^0\right).$$

Therefore, from (3.80) it follows that (3.78) is hold if and only if (3.79) is hold. The Theorem 3.21 in the case of the infinite number of edges in Newton's majorante is proved.

Suppose that the diagram includes the ray $[P_N;\infty)$, of course, with an angular coefficient $\alpha_0=A$. Then, on the one hand, the maximum term is bounded and, due to (3.77), (3.78) is hold, and on the other hand, and (3.79) is hold. The Theorem 3.21 is completly proved.

Using the connections between the maximum modulus and the maximum term, we can find the conditions for the sequence (λ_n), under which in (3.78) we can put $\ln\mu(\sigma)$ instead of $\ln M(\sigma)$.

Corollary 3.13. *Let the entire Dirichlet series (3.1) have the finite R-order $\rho_R>0$ and $\ln n=O(\lambda_n)$, when $n\to\infty$. Then, in order that the relation (3.76) to hold, it is necessary and in the case, when the sequence $(\kappa_n(F))$ is nondecreasing it is sufficient, that*

$$\sum_{n=1}^{\infty}\left(\lambda_n-\lambda_{n-1}\right)\left|a_n\right|^{\rho_R/\lambda_n}<+\infty. \qquad (3.82)$$

Indeed, it follows from the condition $\ln n=O(\lambda_n)\,(n\to\infty)$ that $\tau_0<+\infty$, and by the Corollary 3.6 from the Theorem 3.9, the inequality (3.32) takes place. From (3.32) and the Cauchy’s inequality follow, that (3.76) is hold if and only if

$$\int_0^{\infty} \frac{\ln M(\sigma, F)}{\exp(\rho_R \sigma)} d\sigma < +\infty, \tag{3.83}$$

i. e. (3.78) is hold with $\beta(\sigma) = \exp\{\rho_R \sigma\}$. In this case

$$B(x) = \int_x^{\infty} \int_t^{\infty} \frac{d\sigma}{\exp(\rho_R \sigma)} dt = \frac{1}{\rho_R^2 \exp\{\rho_R x\}}.$$

For entire Dirichlet series $\sigma_a = \alpha_0 = +\infty$ (see the Theorem 3.3). Therefore, by the Theorem 3.21, the ratios (3.83) are hold if and only if

$$\sum_{n=1}^{\infty} (\lambda_n - \lambda_{n-1}) \exp\left\{ -\frac{\rho_R}{\lambda_n} \ln \frac{1}{a_n^0} \right\} < +\infty. \tag{3.84}$$

But $|a_n| \le a_n^0$. Therefore from (3.84) follows (3. 82). In the case when the sequence $(\kappa_n(F))$ is non-decreasing, we have $|a_n| = a_n^0$, so the ratio (3.84) and (3.82) are equivalent. The Corollary 3.13 is proved.

Since the definition of the order of the entire function and the logarithmic R-order $p_R > 1$ is defined by the convergence of the integral

$$\int_1^{\infty} \frac{\ln M(\sigma, F)}{\sigma^{p_R + 1}} d\sigma < +\infty. \tag{3.85}$$

Corollary 3.14. *Let the entire Dirichlet series (3.1) have the finite logarithmic R-order $p_R > 1$ and either $\ln n = o\left(\lambda_n^{p_R/(p_R - 1)}\right)$, when $n \to \infty$, or (3.34) to hold. In order that (3.85) to hold, it is necessary and in the case when the sequence $(\kappa_n(F))$ is non-decreasing it is sufficient, that*

$$\sum_{n=1}^{\infty} (\lambda_n - \lambda_{n-1}) \left(\frac{1}{\lambda_n} \ln \frac{1}{|a_n|} \right)^{1 - p_R} < +\infty. \tag{3.86}$$

In fact, if (3.34) is hold, then from (3.35) and Cauchy's inequality easily follows, that (3.85) takes place if and only if

$$\int_{1}^{\infty} \frac{\ln \mu(\sigma, F)}{\sigma^{p_R+1}} d\sigma < +\infty .$$

(3.87)

In the case, when $\ln n = o\left(\lambda_n^{p_R/(p_R-1)}\right)$, $n \to \infty$, we obtain the same conclusion on the basis of the Corollary 3.8 from the Theorem 3.11 with $\Phi(x) = x^{p_R}$ for all sufficiently large x.

If we take $\beta(\sigma) = \sigma^{p_R+1}$, then

$$B(x) = \frac{1}{p_R(p_R-1)} x^{1-p_R}$$

and by the Theorem 3.2, the ratio (3.17) is hold if and only if

$$\sum_{n=1}^{\infty} (\lambda_n - \lambda_{n-1}) \left(\frac{1}{\lambda_n} \ln \frac{1}{|a_n^0|} \right)^{1-p_R} < +\infty .$$

Further proof of the Corollary 2 is the same as the Corollary 1.

Let's consider the Dirichlet series, absolutely convergent in the half-plane $\{s : \operatorname{Re} s < 0\}$. For such series of the finite logarithmic R-order $p_R^0 > 0$, the convergence class is defined by the convergence of the integral

$$\int_{-1}^{0} |\sigma|^{p_R^0-1} \ln M(\sigma) d\sigma < +\infty . \qquad (3.88)$$

Corollary 3.15. *Let the absolutely convergent in the half-plane* $\{s : \operatorname{Re} s < 0\}$ *Dirichlet series (3.1) have the finite logarithmic* R*-order* $p_R^0 > 0$ *and either*

$$\overline{\lim_{n\to\infty}} \frac{\ln \ln n}{\ln \lambda_n} < \frac{p_R^0}{p_R^0 + 1} , \qquad (3.89)$$

or $\ln n = o\left(\ln^+ |a_n|\right)$, $n \to \infty$. *In order that the relation (3.88) to hold, it is necessary, and in the case, when the sequence* $(\kappa_n(F))$ *is non-decreasing, it is sufficient that*

$$\sum_{n=1}^{\infty}(\lambda_n-\lambda_{n-1})\left(\frac{\ln^+|a_n|}{\lambda_n}\right)^{p_R^0+1}<+\infty. \qquad (3.90)$$

In fact, if the condition (3.89) takes place, then firstly $\ln n=o(\lambda_n)$, $n\to\infty$, and secondly $\ln n=\lambda_n^{\alpha}\,(n\ge n_0)$, where $\alpha<\frac{p_R^0}{p_R^0+1}$, i. e. (3.56) is hold with $\gamma(x)=x^{\alpha-1}$ and $h_0=1$. Therefore, by the Corollary 3.1 from the Theorem 3.3 $\sigma_a=\alpha_0=0$, and by the Theorem 3.15, the inequality (3.57) is true, from which in this case for all $\sigma<0$ we have

$$\ln M(\sigma)\le\ln\mu\left(\frac{\sigma}{1+\varepsilon}\right)+K_1(\varepsilon)|\sigma|^{1-1/(1-\alpha)},\, K_1(\varepsilon)=const,\, \varepsilon>0,$$

i.e.

$$\int_{-1}^{0}|\sigma|^{p_R^0-1}\ln M(\sigma)\,d\sigma\le\int_{-1}^{0}|\sigma|^{p_R^0-1}\ln\mu\left(\frac{\sigma}{1+\varepsilon}\right)d\sigma+K_1(\varepsilon)\int_{-1}^{0}|\sigma|^{p_R^0-1/(1-\alpha)}\,d\sigma.$$

Since $\alpha<\frac{p_R^0}{p_R^0+1}$, then the last integral in this inequality is convergent, and therefore, taking into account the Cauchy’s inequality , we see that the ratio (3.88) is equivalent to the ratio

$$\int_{-1}^{0}|\sigma|^{p_R^0-1}\ln\mu(\sigma)\,d\sigma<+\infty. \qquad (3.91)$$

If $\ln n=o\left(\ln^+|a_n|\right)$, $n\to\infty$, then by the Corollary 4 $\sigma_a=\alpha_0=0$, and by the Corollary 3.9 from the Theorem 3.28 we have (3.42) with $h_0=0$, whence the equivalence of the relations (3.88) and (3.91) again follows.

Let's choose $\beta(\sigma)=|\sigma|^{1-p_R^0}$. If $p_R^0\ge1$, then the function β is non-decreasing on $(-\infty;0)$ and

$$B(x)=\frac{1}{p_R^0\left(p_R^0+1\right)}|x|^{1+p_R^0}. \tag{3.92}$$

Therefore, by the Theorem 3.21, the ratio (3.91) is hold if and only if

$$\sum_{n=1}^{\infty}(\lambda_n-\lambda_{n-1})\left(\frac{1}{\lambda_n}\ln\frac{1}{a_n^0}\right)^{p_R^0+1}<+\infty. \tag{3.93}$$

Since $\left|\frac{1}{\lambda_n}\ln\frac{1}{a_n^0}\right|=\left|\frac{1}{\lambda_n}\ln\frac{1}{a_n^0}\right|$ and $\ln^+|a_n|\le\ln a_n^0$, then from (3. 93) follows (3.90). In the case when the sequence is non-decreasing, we obtain $(\kappa_n(F))\,|a_n|=a_n^0$ and therefore the relations (3.93) and (3.90) equivalent. The Collorary 3 for $p_R^0\ge1$ is completely proved.

If $0<p_R^0<1$, the function β is decreasing and we can not refer to the Theorem 3.2. But from (3.92) it is clear that in this case the function $B^{1/p}$, when $p=p_R^0+1$, is convex. By the Theorem 3.1 the conclusion of the Theorem 3.21 is true and, hence, the conclusion of the Corollary 3. 15 takes place.

Suppose that the absolutely convergent in the half-plane $\{s:\operatorname{Re}s<0\}$ the Dirichlet series (3.1) has a finite *R*-order $\rho_R^0>0$, and the convergence class is defined by the convergence of the integral

$$\int_1^{\infty}\frac{\ln M(\sigma,F)}{|\sigma|^2\exp\left(\rho_R^0/|\sigma|\right)}d\sigma<+\infty\ . \tag{3.94}$$

Corollary 3. 16. *Let* $\ln n=O(\ln\lambda_n),\ n\to\infty$ *and the Dirichlet series (3.1) have a zero abscissa of absolute convergence. In order that the relation (3.48) to hold, it is necessary, and in the case, when the sequence* $(\kappa_n(F))$ *is non-decreasing, it is sufficient that*

$$\sum_{n=1}^{\infty}(\lambda_n-\lambda_{n-1})\left(\frac{\ln^+|a_n|}{\lambda_n}\right)^2\exp\left\{-\frac{\rho_R^0\lambda_n}{\ln^+|a_n|}\right\}<+\infty\ . \tag{3.95}$$

It follows that (3.94) takes place if and only if

$$\int_1^{\infty} \frac{\ln \mu(\sigma, F)}{|\sigma|^2 \exp\left(\rho_R^0 / |\sigma|\right)} d\sigma < +\infty, \tag{3.96}$$

i. e. (3.78) is hold with $\beta(\sigma) = |\sigma|^2 \exp\left(\rho_R^0 / |\sigma|\right)$. It is easy to see, that the function β is increasing on $\left[-\rho_R^0 / 2; 0\right)$ and

$$B(x) = \frac{1}{\rho_R^0} \int_0^x \frac{dt}{\exp\left\{\rho_R^0 / |t|\right\}} = \frac{1}{\rho_R^0} \int_{1/|x|}^{+\infty} \frac{dt}{t^2 \exp\left\{\rho_R^0 t\right\}}.$$

Using the L'Hopital's rules, we obtain

$$\lim_{y \to +\infty} \frac{\int_y^{+\infty} t^{-2} \exp\{-\rho t\} dt}{y^{-2} \exp\{-\rho y\}} = \lim_{y \to +\infty} \frac{y^{-2} \exp\{-\rho y\}}{2y^{-3} \exp\{-\rho y\} + \rho y^{-2} \exp\{-\rho y\}} = \frac{1}{\rho}.$$

Therefore

$$B(x) = \frac{1 + o(1)}{\left(\rho_R^0\right)^2} |x|^2 \exp\left\{-\frac{\rho_R^0}{|x|}\right\}, \quad x \to 0,$$

and by the Theorem 3.21, the relation (3.96) is hold if and only if, when

$$\sum_{n=1}^{\infty} (\lambda_n - \lambda_{n-1}) \left|\frac{1}{\lambda_n} \ln \frac{1}{a_n^0}\right|^2 \exp\left\{-\frac{\rho_R^0}{\left|\frac{1}{\lambda_n} \ln \frac{1}{a_n^0}\right|}\right\} < +\infty.$$

The further proof of the Collorary 3.16 is the same as the proof of the Collorary 3.15.

Control questions

1. Give the formulas for calculating the convergence abscissa of the Dirichlet series.

2. Formulate the three-lines theorem.
3. Formulate the definition of the maximum term and the central index of the Dirichlet series.
4. Give the formulas for calculating the *R*-order and *R*-type of the Dirichlet series.
5. Give the definition of the order and type of absolutely convergent in the half-plane Dirichlet series.
6. Formulate the definition of the convergence class of the entire function.

Examples of problem solving

1. Find the abscissas of convergence and the absolute convergence of the series

$$\sum_{n=1}^{\infty} n^{-2n} e^{sn\ln n}.$$

If in our case $\tau = \overline{\lim_{n\to\infty}} \frac{\ln n}{\lambda_n} = \overline{\lim_{n\to\infty}} \frac{\ln n}{n\ln n} = \lim_{n\to\infty} \frac{1}{n} = 0$, then

$$\sigma_a = \sigma_ç = -\overline{\lim_{n\to\infty}} \frac{\ln|a_n|}{\lambda_n} = -\overline{\lim_{n\to\infty}} \frac{\ln\left(n^{-2n}\right)}{n\ln n} = \overline{\lim_{n\to\infty}} \frac{2n\ln n}{n\ln n} = 2.$$

Tasks for independent work

Find the abscissas of convergence and the absolute convergence of the series:

a) $\sum\limits_{n=0}^{\infty} e^{-n^3} e^{sn^2}$. Answer: $\sigma_ç = \sigma_a = +\infty$;

b) $\sum\limits_{n=2}^{\infty} (-1)^n \frac{1}{n} e^{s\ln\ln n}$. Answer: $\sigma_ç = +\infty, \sigma_a = -1$;

c) $\sum_{n=2}^{\infty}(-1)^n \frac{1}{\sqrt{n}} e^{s\ln\ln n}$. Answer: $\sigma_{ç}=+\infty, \sigma_a=-\infty$;

d) $\sum_{n=1}^{\infty}(-1)^n e^{s\ln n}$. Answer: $\sigma_{ç}=0, \sigma_a=-1$;

e) $\sum_{n=1}^{\infty} e^{e^n} e^{sn^2}$. Answer: $\sigma_{ç}=\sigma_a=-\infty$;

f) $\sum_{n=1}^{\infty} n^{3n} e^{sn\ln n}$. Answer: $\sigma_{ç}=\sigma_a=-3$;

g) $\sum_{n=1}^{\infty} \frac{1}{n} e^{sn}$. Answer: $\sigma_{ç}=\sigma_a=0$;

h) $\sum_{n=1}^{\infty} e^{-2n^2} e^{sn^2}$. Answer: $\sigma_{ç}=\sigma_a=2$;

i) $\sum_{n=1}^{\infty}\left(1+n^2\right)^{-2} e^{sn}$. Answer: $\sigma_{ç}=\sigma_a=0$.

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