

Drohobych Ivan Franko State Pedagogical University

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**SCIENTIFIC FOUNDATIONS
OF THE SCHOOL COURSE OF MATHEMATICS.**

Part I

TEXTS OF LECTURES

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2026

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Ольга КУТНЯК

**НАУКОВІ ОСНОВИ
ШКІЛЬНОГО КУРСУ МАТЕМАТИКИ.**

Частина I

ТЕКСТИ ЛЕКЦІЙ

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Scientific Foundations of the School Course of Mathematics.

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The texts of lectures are edited in accordance with the programme of the course "Scientific Foundations of the School Course of Mathematics" for the preparation of applicants for the second (Master's) level of higher education in the field of knowledge *A Education*, subject specialty A4.04 "*Secondary Education (Mathematics)*", approved by the Scientific and Methodological Council of Drohobych Ivan Franko State Pedagogical University (protocol № 8 dated 27. 08. 2026). They cover the methodological foundations of Mathematics; the relationship between set theory and school Mathematics; mapping and functions in the school course of Mathematics; polynomials, equations and inequalities.

The manual is recommended for training the specialists of the second (Master's) level of higher education in the subject specialty A4.04 "*Secondary education (Mathematics)*" of full-time and part-time forms of study.

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Наукові основи шкільного курсу математики. Ч. І: тексти лекцій [для здобувачів другого (магістерського) рівня вищої освіти спеціальності А4.04 Середня освіта (Математика)] / уклад. О. А. Кутняк. Дрогобич: Дрогобицький державний педагогічний університет імені Івана Франка, 2026. 64 с.

Тексти лекцій укладені відповідно до програми курсу «Наукові основи шкільного курсу математики» для підготовки здобувачів другого (магістерського) рівня вищої освіти галузі знань *А Освіта* предметної спеціальності *А4.04 «Середня освіта (Математика)»*, затвердженої науково-методичною радою Дрогобицького державного педагогічного університету імені Івана Франка (протокол № 8 від 27 серпня 2025 р.). У них висвітлено методологічні основи математики; зв'язок теорії множин та шкільної математики; відображення і функції у шкільному курсі математики; многочлени, рівняння та нерівності.

Посібник рекомендований для підготовки фахівців другого (магістерського) рівня вищої освіти предметної спеціальності *А4.04 «Середня освіта (Математика)»* денної та заочної форм навчання.

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Preface

The academic discipline "Scientific Foundations of the School Course of Mathematic" is intended to improve the mathematical and methodological training of the future teachers of Mathematics. It is aimed for understanding the content of school Mathematics from the standpoint of modern Higher Mathematics and the Teaching Methods of its and involves not only the mastering theoretical provisions, but also developing the ability of future teachers to analyze school mathematical concepts, establish their scientific foundations, and methodically correctly present them in the educational process.

The texts of lectures cover the following course topics: methodological foundations of Mathematics; set theory and school Mathematics; mappings in the school course of mathematics; functions in the school course of Mathematics; polynomials, equations and inequalities. The theoretical material is illustrated with examples.

The manual will be useful for students of mathematical specialties of pedagogical universities, teachers and senior students of secondary education institution.

Передмова

Навчальна дисципліна «*Наукові основи шкільного курсу математики*» призначена для удосконалення математичної та методичної підготовки майбутніх учителів математики. Вона спрямована на осмислення змісту шкільної математики з позицій сучасної вищої математики та методики її навчання і передбачає не лише засвоєння теоретичних положень, а й формування здатності майбутніх учителів аналізувати шкільні математичні поняття, встановлювати їх наукові підґрунтя та методично коректно представляти їх у навчальному процесі.

У текстах лекцій розглянуто наступні теми курсу: методологічні основи математики; теорія множин і шкільна математика; відображення у шкільному курсі математики; функції в шкільному курсі математики; многочлени, рівняння та нерівності. Теоретичний матеріал ілюструється на прикладах.

Посібник буде корисним для студентів математичних спеціальностей педагогічних університетів, учителів та учнів старших класів закладів загальної середньої освіти.

§ 1. Methodological foundations of Mathematics

1.1. *The subject of Mathematics and its characteristic features*

Methodology is a system of principles, methods, rules and ways of organizing the scientific knowledge or practical activity. It determines the research strategy, ensuring the receipt of objective information. The purpose of methodology is to organize the activity to achieve the true knowledge, create new theories or objects.

Mathematics, like the other sciences, studies the surrounding world, objects of this world and the relationships between them. However, unlike the natural sciences, it studies the forms and relationships of the material world, abstracting from their content. The characteristic features of mathematical science include [6], [10]:

- 1) Mathematics studies abstracted properties of objects – numbers, not sets of objects, geometric figures, not real bodies;
- 2) the main method of obtaining the mathematical results is the logical conclusion, which does not require the experimental verification;
- 3) abstractions arising in Mathematics develop from abstractions that directly generalize the properties of real objects to abstractions of such a high level as topological spaces, general algebraic systems, algorithms, etc;
- 4) Mathematics has the property of universal application in the other fields;
- 5) Mathematics occupies a special position in the system of sciences – it cannot be attributed to either the humanities or the natural sciences. It provides the basic concepts that are used in almost all sciences. Among these concepts are "set", "structure", "system", "isomorphism", etc.

1.2. *The main stages of the development of Mathematics*

The main stages of the development of Mathematics include [5], [6]:

- the origin of Mathematics (deep antiquity - 5th century BC);

- Mathematics of the constant quantities (6th-5th century BC - late 16th century);
- Mathematics of the variable quantities (late 16th - mid-18th century);
- the modern period (from the mid-18th century to the present day).

At each stage, new branches and tools were formed that contributed to the development of science in general.

In the period of its origin (deep antiquity - 5th century BC), mathematics emerged as a result of practical human needs: calculation, measurement, counting. During this period, the foundations of arithmetic and geometry were developed. The beginning of the development of mathematics is associated with the first civilizations (for example, in Babylon and Ancient Egypt).

During the period of Mathematics of the constant quantities (6th-5th centuries BC – late 16th century), the basic theories concerning constant quantities were formed. **Ancient Greece** is considered as the birthplace of Mathematics, where the first mathematicians, such as Thales of Miletus, Pythagoras and Euclid, appeared. At this time, the first theoretical treatises were created, such as Euclid's "Principles", which became the basis of school Mathematics. Works on geometry, algebra and arithmetic were created to solve practical problems in surveying, architecture and astronomy.

The period of the variable mathematics (late 16th – mid-18th centuries) is characterized by the emergence of analytical geometry and differential and integral calculus. Prominent scientists of this period were René Descartes (founder of analytical geometry), Isaac Newton, and Gottfried Leibniz, who independently created differential and integral calculus. It became possible to study motion, transformations, and to investigate functions and their properties, which became a powerful tool for the development of physics, mechanics, and other sciences.

In the modern period of the development of Mathematics (from the middle of the 18th century to the present), the rapid development of previously established branches of Mathematics continues. New sections such as mathematical logic, set theory, group theory, algebraic topology, functional analysis, probability theory, digital mathematics, machine learning are emerging. Applied fields are actively developing: applied mathematics, mathematical modeling, operations research, automatic control

theory. Modern mathematics is closely related to computers, which have become an indispensable tool for complex calculations and research.

1.3. Mathematical methods of cognition. The concepts of number, figure and set as examples of mathematical models

A key aspect of human cognitive activity is the study of the properties and interactions of objects of reality with the aim of their further practical application. Mathematics has formed a system of methods for the theoretical understanding of spatial and qualitative characteristics of the world. Among them, a special place is occupied by a mathematical model – an approximate description of any class of phenomena of the external world, expressed using the mathematical symbols. The construction of mathematical models is a method of cognition of the external world, prediction of phenomena and management of various processes. The method of mathematical modeling is widely used in various sciences: Physics, Chemistry, Biology, Economics, Sociology [6], [10].

The process of forming mathematical models inevitably involves ignoring certain aspects of reality, which causes a certain discrepancy between the model and the object of research. Only a comparison with reality of the results obtained by studying the model allows us to draw the conclusions about the quality of this model, the limits of its application. Each model can be used only under certain conditions. The foundation of mathematical modeling is a specific process of abstraction, the most common forms of which in Mathematics are idealization and abstraction of identification (generalization).

Abstraction of identification begins with the establishment of an equivalence relation in the set under investigation. When establishing an equivalence relation in a certain set, equivalent objects are identified by some property, which is abstracted from the rest of the properties of these objects and becomes an independent abstract concept. Thus, the equivalence relation of sets unites into one class all finite sets between which a bijective correspondence can be established. As a result of this identification, from

sets belonging to the same equivalence class, their general property characterizing this class is abstracted, which is inherent to all sets of this class and is not inherent to any other sets. This property is an independent concept of a natural number, which expresses the number of sets of a given class [6], [10].

Also, when constructing mathematical models of reality, such a specific abstraction technique as idealization is widely used, which consists in forming concepts that are endowed not only with real, but also with hypothetical properties that are not inherent in prototype objects. The creation of such idealized constructs is aimed at simplifying the analysis of complex natural phenomena. The fundamental concepts of Mathematics, such as "geometric point" (which has no physical dimensions), "line" or "plane", are the classic examples of idealizations. Despite the absence of their exact analogues in the material world, the construction of a holistic geometry is impossible without these categories. The next stage of scientific knowledge is the application of the abstraction of identification: by generalizing the specific features of individual idealized figures, universal concepts are formed (for example, the general categories "sphere" or "triangle").

1.4. Axiomatic method in Mathematics

The axiomatic method is a special way of constructing a scientific theory, it is based on several initial propositions (**axioms**) that are accepted without the proof. All the other statements of the theory (theorems) are derived from these axioms in a purely logical way. In order for an axiomatic system to be considered perfect, it must meet three criteria:

1) *consistency*: it is impossible to derive a statement and its negation from the axioms at the same time. This is the most important requirement - a contradictory system has no value.

2) *independence*: none of the axioms can be derived from the others. This makes the system concise.

3) *completeness*: any true statement within this system can be proven based on its axioms.

The first example of the axiomatic method is the «Euclid's Elements» (c. 300 BC). Euclid attempted to derive all of the Geometry from a few obvious postulates. Although his system was incomplete from the point of view of modern logic, it remained the ideal of science for two millennia.

A qualitative leap occurred at the turn of the 19th and 20th centuries thanks to the works of David Hilbert. He developed the formal axiomatics, where the objects (points, lines) are considered not as the physical entities, but as abstractions, the properties of which are completely determined by the axioms. This allowed Mathematics to go beyond the boundaries of clarity and work with multidimensional spaces and abstract structures. Let us give an example of the axiomatic construction of Geometry, described by D. Hilbert in the book "Fundamentals of Geometry" (1899). [6], [3].

The first group of axioms: the axioms of belonging.

The basic relation is “*to belong*”.

I_1 . For any two points A, B , there is a line a , that belongs to each of these two points A, B .

I_2 . For any two points A, B , there is at most one line that belongs to each of these points A, B .

I_3 . There are at least two points on a line. There are at least three points that do not lie on the same line.

I_4 . For any three points A, B, C , that do not lie on the same line, there is a plane α , that belongs to each of these three points A, B, C . For any plane, there is always a point that belongs to it.

I_5 . For any three points A, B, C , which do not lie on the same line, there is at most one plane that belongs to these points.

I_6 . If two points A, B of a line lie in the plane α , then any point of the line α lies in the plane.

I_7 . If two planes α and β have a common point A , then they have at least one other common point B .

I_8 . There are at least four points that do not lie on the same plane.

The axioms I_{1-3} are called the planar axioms of the group I , and I_{4-8} are called the spatial axioms of the group I .

The second group of axioms

The main concept is “between”.

II_1 . If a point B lies between the points A and C , then A, B, C are three distinct points on the line and B lies also between C and A .

II_2 . For any points A and C on the line AC there is at least one point B such that the point C lies between A and B .

II_3 . Among any three points on a line, there is at least one point that lies between the other two.

The third group of axioms: the axioms of equality and motion

The main concept is “equal”.

III_1 . If A, B there are two points on a line a and A' is point on the same line or on another line a' , then it is always possible to find a point B' that lies on a given side of the line a' from the point A' and such that the segment AB is equal to the segment $A'B'$.

III_2 . If a segment $A'B'$ and a segment $A''B''$ are equal to the same segment AB , then the segment $A'B'$ is also equal to the segment $A''B''$; in other words, if two segments are equal to a third, then they are equal to each other.

III_3 . Let AB i BC are the two segments of the line a , that do not have any common point, and let $A'B'$ i $B'C'$ are two segments of the same line or the other line a' , which also do not have any common point; if $AB \equiv A'B'$ and $BC \equiv B'C'$, then $AC \equiv A'C'$.

III_4 . Let the angle $\angle(h, k)$ in the plane α and a line a' in the plane α' be given, and the side relatively a' of the plane α' be defined. Let h' denote the ray of the line a' , which

comes out from the point O' . Then in the plane α' there exists one and only one ray k' such that the angle (h, k) is equal to the angle (h', k') and at the same time all the interior points of the angle (h', k') lie on the given side of a' . This relation between the angles is denoted as follows: $\angle(h, k) \equiv \angle(h', k')$.

III_5 . If for two triangles ABC and $A'B'C'$ the equalities $AB \equiv A'B'$, $AC \equiv A'C'$, $\angle BAC \equiv \angle B'A'C'$ are hold, then the equality $\angle ABC \equiv \angle A'B'C'$ also is hold.

The fourth group of axioms: the axiom of parallels

IV. (Euclid's axiom). Let a be an arbitrary line and A be a point that lies outside the line, then in the plane defined by the point A and the line a , there exists at most one line that passes through the point A and does not intersect the line a .

The fifth group of axioms: continuity axioms

V_1 . (measurement axiom or Archimedes' axiom). Let AB and CD be two arbitrary segments, then there are a finite number of points $A_1, A_2, A_3, \dots, A_n$ on the line AB such that the segments $AA_1, A_1A_2, \dots, A_{n-1}A_n$ are equal to the segment CD and the point B lies between A and A_n .

V_2 . (linear completeness axiom). The points of the line form a system that, while preserving the linear order, the first axiom of equality and the axiom of Archimedes (i.e. the axiom I_{1-2}, II, III_1, V_1), does not admit of any extension.

Dedekind's axiom. If all points on a line are divided into two classes such that:

- 1) each point belongs to one and only one class and each class contains points;
- 2) each point of the first class precedes each point of the second,

then either in the first class there is a point preceded by all other points of the first class, or in the second class there is a point which precede all the other points of the second class.

Control questions

1. What does the methodology study?
2. Name the characteristic features of Mathematics as a science?
3. Briefly describe the main stages of the development of Mathematics.
4. What is a mathematical model?
5. Describe such abstraction techniques as abstraction, identification (generalization) and idealization.
6. What is the essence of the axiomatic method in Mathematics?
7. Give the examples of the axiomatic construction of geometry.

§ 2. Set theory and school Mathematics.

2.1. "Naive" and axiomatic set theories

Naive and axiomatic set theory are two different approaches to the study of sets. Naive set theory uses an intuitive, "natural" understanding of sets and their properties, while axiomatic set theory formalizes the concept of sets using a system of axioms that eliminates the paradoxes that arise in naive theory.

Cantor's naive definition of a set (or, as it is also called, the intuitive definition) describes a set as an arbitrary collection of certain objects that can be distinguished from each other and that are imagined as a single whole. In other words, it is a collection of objects that are considered as a single whole, without delving into formal axioms [2], [6].

Intuitive definition – a set is considered as an arbitrary collection of objects united by a certain attribute. There are no clear rules that determine which operations on sets are permissible. Due to the lack of formalization, naive set theory leads to the logical contradictions, such as Russell's paradox.

In axiomatic set theory the concept of a set is defined by a system of axioms that are accepted without the proof. The axiomatic set theory was developed to overcome the paradoxes of naive set theory. The most common axiomatic set theory is the Zermelo-Frenkel system. Here are its axioms:

1. *Extensional axiom.* Two sets are equal if and only if they have the same elements.
2. *Empty set axiom.* There is a set that does not contain any elements.
3. *Pair axiom.* For any two sets, there is a set that contains exactly them.
4. *The axiom of union.* For two sets, there exists a third set that includes all elements of both, and only them.
5. *The axiom of the power set.* For any set, there exists a set of all its subsets.
6. *Axiom of selection (subset scheme).* For any set and property, a subset can be selected.

7. *Axiom of substitution (substitution scheme).* The image of a set under a functionally given mapping is a set (i.e., if each element of the set is assigned a different object (according to a rule), then all these new objects also form a set).

8. *Axiom of regularity (foundation).* Sets cannot contain themselves or form cycles.

9. *Axiom of infinity.* There exists an infinite set.

10. *Axiom of choice.* For any family of non-empty, non-intersecting sets, there exists a set that has exactly one element in common with each set in the given family, even if the family has infinitely many sets and the rule for choosing an element from each set is undefined.

2.2. Numerical sets of school Mathematics

In the school Mathematics course, the theory of real numbers is not fully study. The proofs found in this theory, and even the definition of a real number, are quite complex and use a number of ideas that are far from the school course. However, the term "real number" inevitably appears in the lessons of Mathematics. In particular, students are told about rational and irrational numbers (the precise definition of which is also not formulated).

We come to rational numbers by gradually generalizing the concept of the number. The first numbers that we get to know in the younger grades of school are the natural numbers: 1, 2, 3,... The set of all natural numbers is infinite. In this set, two operations are always performed: "+" and "•". In this case, the sum and product of two natural numbers are again natural numbers. As for the inverse operations "-" and ":", they are not always performed on the set of the natural numbers. For example, 5-9 and 2:3 cannot be calculated without going beyond the set. In order for these operations to be performed always, the new numbers are introduced – fractional and negative. As a result, we come to rational numbers.

Usually, the fractional and negative numbers are not introduced at the same time. In school, the fractional numbers are introduced first (more precisely, positive

fractional numbers: $\frac{1}{2}, \frac{3}{4}, \frac{1}{10}, \frac{3}{2}$ etc.). As a result of the introduction of these new operations, the ":" of one natural number to the second (or even one fractional to the second) is always performed.

Next, 0 and negative rational numbers are introduced. As a result, we get the set of all rational numbers. 4 arithmetic operations are performed on this set (except for the ":" on 0). Another way is also possible: first, 0 and negative numbers are added (getting Z), then then fractional numbers are added. There is no clear definition of rational numbers in the school curriculum. The axiomatic way of constructing rational numbers remains. In this case, it is known what a natural number is, then the term "rational number" is introduced without a definition and the properties of rational numbers are formulated.

The set Q of all rational numbers has the following 4 groups of properties:

1) $\forall a, b \in Q$ their sum $a + b$ is defined and

$$a + b = b + a \text{ (commutativity);}$$

$$(a + b) + c = a + (b + c) \text{ (associativity);}$$

$\exists$ the number 0 that $a + 0 = a \forall$ rational a ;

$\forall a, b \in Q \exists! x \in Q$ that is the solution of the equation $b + x = a$. The number x is called the difference a and b , $x = a - b$;

$$0 - a = -a.$$

2) the set Q contains the natural numbers $Q \supset N$. The rational numbers, represented as the difference of two natural numbers, are called the integers (Z).

3) for $\forall a, b \in Q$ their product is $a \cdot b$, is defined and

$$ab = ba \text{ (commutativity);}$$

$$(ab)c = a(bc) \text{ (associativity);}$$

$$a(b + c) = ab + ac \text{ (distributivity);}$$

$$\forall a \in Q \quad 1 \cdot a = a;$$

$\forall a, b \in Q, b \neq 0, \exists! x \in Q$, that $bx = a$, x is called the *quotient* of the numbers a and b , and denote $x = \frac{a}{b}$ (division operation);

4) any rational number is represented as a fraction $\frac{a}{b}$, $a, b \in \mathbb{Z}$, $b \neq 0$.

These four groups of properties completely characterize the set of rational numbers (proving this is beyond the scope of the school curriculum). The fulfillment of these properties can be taken as a definition of the set of all rational numbers.

However, it is not enough only with the rational numbers. You have to introduce the numbers that are not rational and are called irrational. Together, rational and irrational numbers are called real numbers.

The arithmetic reason for introducing the irrational numbers: the quadratic equations and equations of the higher degrees with the rational (and even integer) coefficients cannot always be solved while remaining in the set $\mathbb{Q}$ (for example, $x^2 = a$, $x^3 = a$ etc., $a > 0$, $a \in \mathbb{Q}$).

In the school algebra books, there is a proof of the fact that $\sqrt{2}$ (i.e., the positive root of the equation $x^2 = 2$) is irrational, or, more precisely, that there is no rational number whose square is 2. Even extracting (arithmetic) roots requires the introduction of irrational numbers.

The set of rational numbers can be defined as follows (after describing the set of all real numbers): irrational are real numbers that are not rational, that is, that cannot be represented as two integers.

The geometric reason for introducing irrational numbers: there are not enough rational numbers to measure the segments.

An irrational number is a number that is written as an infinite non periodic decimal fraction (an incorrect definition, since it is incomplete and does not describe operations on such numbers).

The set $\mathbb{R}$ of all real numbers has the following 5 groups of properties:

- 1) 1) the set $\mathbb{R}$ contains all rational numbers $\mathbb{R} \supset \mathbb{Q}$;
- 2) see property 1, we will have the same for the set $\mathbb{R}$;
- 3) see property 3, we will have the same for the set $\mathbb{R}$;
- 4) $\mathbb{R}$ are divided into the positive and negative numbers. In this case, the sum and product of positive numbers is again a positive number. This makes it possible to

introduce the inequalities in the set R : by the definition $a > b$ (or $b < a$), if $a - b$ – positive.

Furthermore, if a is negative, then $-a$ is positive. $\forall a > 0, a \in R$ we find $r > 0, r \in Q$, that $r < a$.

5) in the set R every bounded monotone sequence has a limit.

The fulfillment of properties 1) – 5) can be taken as the definition of the set of all real numbers.

2.3. Point sets

In the school Geometry the geometric figures are studied, that is, the subsets of the set Π_3 of the points in the three-dimensional space. The coordinate method allows you to specify each point in the three-dimensional space as a set of three real numbers (the coordinates of this point).

Therefore, each geometric figure corresponds to some set of triples of real numbers. Points in the plane are given by two coordinates, therefore, the plane sets correspond to sets consisting of pairs of real numbers. The point sets studied in the school Mathematics are usually given by the tuples of real numbers. For example, the triangle ABC with vertices $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ is given by the tuple of the six numbers $(x_1, y_1, x_2, y_2, x_3, y_3)$. The circle with radius R with the center $A(x_1, y_1)$ is given by the triple of numbers (x_1, y_1, R) etc.

Most often, the geometric figures are defined by their characteristic properties, that is, the properties that the points of these figures possess and that the other points in the space do not possess. For example, the segment AB consists of such points M that $|AM| + |MB| = |AB|$, the disc of radius R with the center O in the plane consists of such points M of this plane, that $|OM| \leq R$ etc.

In these examples, the point sets were given geometrically by the properties of the distances of their points from some fixed points. If a coordinate system is chosen in a space (on a plane), then the geometric figures are given by the certain sets of

equations and inequalities that relate the coordinates of the points of these figures. For example, a straight line is given by the equation $Ax + By + C = 0, A^2 + B^2 \neq 0$.

Specifying of the geometric figures with the equations and inequalities allows to use the apparatus of algebra for solving the geometric problems and, conversely, to provide a geometric illustration of algebraic statements.

2.4. Operations on the sets in school Mathematics

The operations on the sets are the mathematical operations that allow you to create new sets based on one or more existing ones. The most common operations include union ($A \cup B$, all elements in A or B), intersection ($A \cap B$, elements in A and B), difference ($A \setminus B$, elements in A but not in B), and complement (A^c , elements not in A , within the universal set). These operations are fundamental to set theory and are visualized using Venn diagrams, which show the logical relationships between sets using overlapping circles.

In the school course of Mathematics, the sets begin to be studied in the 8th grade during the study of the topic “Square roots. Real numbers” (curriculum “Algebra 8th grade of the New Ukrainian School” based on the model programs of O.S. Ister 2025-2026 academic year). In particular, they study the concepts of a set and its elements, subsets, numerical sets, rational, irrational and real numbers. In the 10th profile grade, they continue to study operations on sets, depict the union and intersection of sets on diagrams or a number line, and illustrate the concept of a subset.

Example 2. 1. Let $A = \{x \in \mathbb{R} : x^2 - 4 \leq 0\}$, $B = \{x \in \mathbb{R} : |x - 1| < 2\}$. Find $A \cap B$, $A \cup B$ and $A \setminus B$.

Solution. Hence,

$$x^2 - 4 \leq 0 \Leftrightarrow (x - 2)(x + 2) \leq 0 \Rightarrow x \in [-2, 2],$$

$$A = [-2, 2].$$

Let's find the set B :

$$|x - 1| < 2 \Leftrightarrow -2 < x - 1 < 2 \Leftrightarrow -1 < x < 3.$$

Thus, $B = (-1, 3)$.

Let's find the intersection of the sets:

$$A \cap B = [-2, 2] \cap (-1, 3) = (-1, 2],$$

their union:

$$A \cup B = [-2, 2] \cup (-1, 3) = [-2, 3),$$

And the difference: $A \setminus B = [-2, 2] \setminus (-1, 3) = [-2, -1]$.

Solving such a problem is a direct application of the operations on the sets (intersection, union, difference). In the school tasks, the sets are replaced by the sets of solutions of the inequalities; the analysis of the intersections of the intervals is a work with the sets in a practical context.

Example 2.2. There are 30 students in a class, 18 of whom study Mathematics, 12 – Physics, and 7 study both Mathematics and Physics. Using the set theory, determine how many students study only Mathematics, only Physics, and at least one of the two subjects.

Solution. Let M be the set of students studying Mathematics and P be the set of students studying Physics. Students studying both subjects: $|M \cap P| = 7$. Then the number of students studying only mathematics: $|M \setminus P| = |M| - |M \cap P| = 18 - 7 = 11$. The number of students studying only physics: $|P \setminus M| = |P| - |M \cap P| = 12 - 7 = 5$. The number of students studying at least one of the two subjects:

$$|M \cup P| = |M| + |P| - |M \cap P| = 18 + 12 - 7 = 23.$$

So, 11 students study only mathematics, 5 students study only physics, and 23 students study at least one of the two subjects.

This example shows how set operations such as union, intersection, and difference are useful for solving real-world problems related to counting and classification.

These examples demonstrate the connection between the set theory and school Mathematics. In geometry, the sets allow, for example, to classify the shapes based on their properties. Venn diagrams can be used to visualize the hierarchy of quadrilaterals, since many of them are subsets of others (for example, every square is a rhombus, but not every rhombus is a square). Let us use these diagrams to present a logical classification of convex quadrilaterals (Fig. 1):

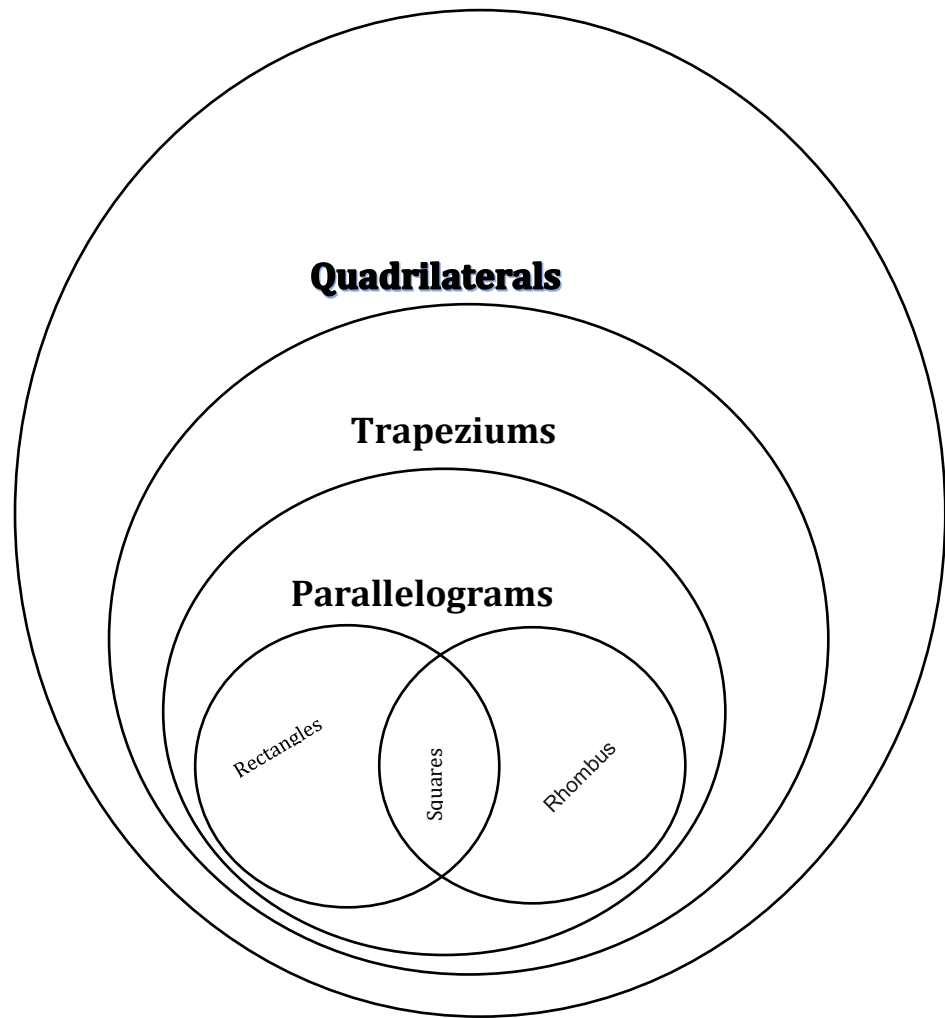


Fig. 1

Thanks to the set theory, abstract thinking becomes a practical tool for analyzing and structuring mathematical knowledge.

2.5. The power of the set. Cardinality above the continuum

The cardinality of a set is the number of elements of the set. For a finite set, this is simply the result of counting its elements. The concept extends to the infinite sets, where cardinality measures their size in terms of countability or uncountability [6].

Two sets have the same cardinality if a one-to-one correspondence (bijection) can be established between their elements. For example, the set of natural numbers and the set of even natural numbers have the same cardinality, since the correspondence $n \leftrightarrow 2n$ can be established. Sets that have the same cardinality as the set of natural numbers are called countable (e.g., $\mathbb{Z}$, $\mathbb{Q}$). Any infinite subset of a countable set is also countable.

A set is defined only by the elements it contains. The order in which the elements are written is not important. Two sets are equal if they consist of the same elements. If the order is important, the concept of an ordered set or tuple is used (for example, $(a, b) \neq (b, a)$).

The cardinality of a set is denoted by $|M|$. The smallest infinite cardinality is $\aleph_0$ (for example, the natural numbers $\mathbb{N}$). The cardinality of a continuum ($\aleph_0 < \aleph_1$) is the cardinality of the set of real numbers $\mathbb{R}$ that is strictly greater than the countable cardinality.

Sets of cardinality above the continuum (e.g., $\aleph_2$) are the sets that contain more elements than real numbers, as proven by the Cantor's theorem (the power set of the set A has a greater cardinality than A), $|P(A)| > |A|$. For example, the set of all functions from $\mathbb{R}$ to $\mathbb{R}$ has the cardinality exceeding the continuum.

Control questions

1. What is the essence of "naive" set theory?
2. Formulate the Zermelo-Frenkel axioms.
3. Formulate the properties of the set $\mathbb{Q}$ of all rational numbers.
4. What is the arithmetic and geometric reason for the emergence of real numbers?
5. Formulate the properties of the set $\mathbb{R}$ of all real numbers.
6. Give the examples of the point sets in school Geometry.
7. What operations are performed on sets in school Mathematics?
8. Illustrate the connection between set theory and school Mathematics with examples.

§ 3. Mappings in the school course of Mathematics

3.1. Mappings and mathematical structures

The concept of mapping is a partial case of the general concept of correspondence between the sets. Namely, a mapping from a set X to a set Y is any functional correspondence between these sets. That is, a mapping from X to Y is a correspondence f such that the image of an arbitrary element $x \in X$ is either an empty set or consists of a single element. A partial case of this concept is the mapping of the set X to the set Y , that is, a functional correspondence between X and Y is defined everywhere. If f is a mapping of X to Y , then the image of each element $x \in X$ consists of a single element $y \in Y$, written $y = f(x)$.

The main concepts related to mappings are the *domain* and the *range* of mappings, the concepts of *injective*, *surjective* and *bijective* mappings, the concepts of *composition* and *inverse mapping*.

Recall that an *injective mapping* is a mapping between elements of two sets in which two different elements of the first set (domain) are never assigned the same element of the second set. That is, a mapping $f: X \rightarrow Y$ is injective if and only if for each y in Y there is at most one (or no) x in X such that $f(x) = y$.

Among the functions studied in the school Mathematics course, injective mappings include, for example, a linear function, a power function with an odd exponent, exponential and logarithmic functions, etc. Non-injective functions include power functions with an even exponent, trigonometric functions, and the function $y = |x|$.

For example, the function $f: \mathbf{R} \rightarrow \mathbf{R}$ defined as $f(x) = x^3$ is injective because for any two real numbers x and x' , if $x^3 = (x')^3$, that's for sure $x = x'$.

Instead, the function $g: \mathbf{R} \rightarrow \mathbf{R}$, defined as $g(x) = x^2$ is not injective because, for example, $g(1) = 1 = g(-1)$.

If, having constructed the graphs of the functions on the coordinate plane, a line is drawn parallel to the abscissa axis, and this line intersects the graph of the function

at one point, then the function is injective. If this line intersects the graph at two or more points, then the function is not injective (Fig. 2):

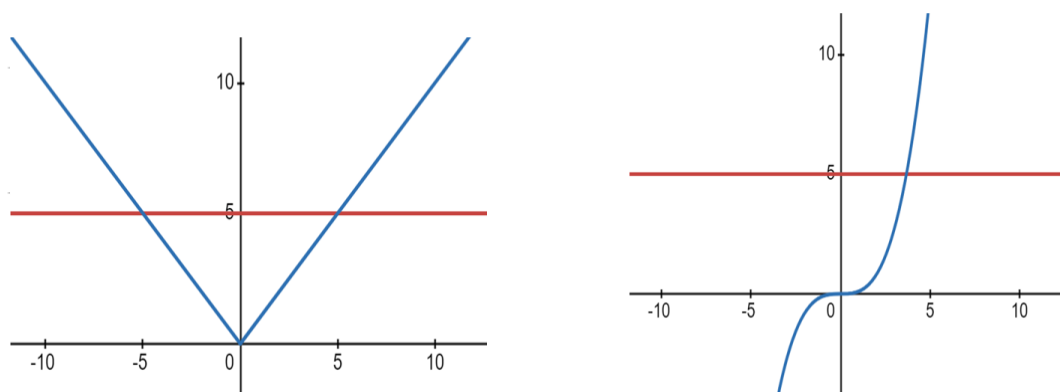


Fig. 2

A mapping $f: X \rightarrow Y$ is called a *surjective* if for every y in Y there exists at least one x in X such that $f(x) = y$.

For example, the function $f: \mathbf{R} \rightarrow \mathbf{R}$, defined as $f(x) = x^2$, is not surjective because there is no real number x such that $x^2 = -1$. But if we define the function $g: \mathbf{R} \rightarrow [0, \infty)$ by the same formula as f , but with a range restricted to nonnegative real numbers only, then the function g will be surjective.

A mapping $f: X \rightarrow Y$ is called *bijective* if it is both injective and surjective.

If there is a bijection $f: X \rightarrow Y$, then the sets X and Y are called *equipotent* and this is written as $|X| = |Y|$. The equipotent relation of a set is an *equivalence*.

A *mathematical structure* is a collection of a set of elements and certain operations or rules that determine how the elements interact with each other. For example, mathematical structures include groups, rings, fields, vector spaces, topological spaces.

A *relation* is a mathematical structure that formalizes the properties of objects and their relationships. For example, the relation of equality, divisibility, parallelism, etc.

3.2. Morphisms of structures

A mapping that preserves mathematical structures is called a *morphism*. It is a fundamental concept used to compare mathematical structures, such as groups, rings, or spaces, by establishing a relationship between their elements that preserves the properties of these structures [6], [7].

A *homomorphism* is a morphism that preserves the algebraic operations, such as addition or multiplication.

An *isomorphism* is a bijective (one-to-one) homomorphism. If two structures are isomorphic, they are mathematically equivalent.

A *homeomorphism* is a morphism that preserves the topological structures. It is a continuous function that has a continuous inverse function.

Recall that a *topological space* is an ordered pair (X, Γ) , where X is a set and Γ is a system of subsets of the set X (they are called open), which satisfies the following conditions [6], [11]:

1. The empty set and the set X belong to Γ .
2. The union of an arbitrary set of sets with Γ also belongs to Γ .
3. The intersection of a finite set of sets with Γ also belongs to Γ . Then the set Γ is called a topology over the set X , and the elements of X are points. Sets in Γ are called open, their complements are closed sets.

3.3. Types of mappings in the school course of Mathematics

In school Mathematics, the pupils mainly study the sets consisting of the numbers or numerical tuples (numerical sets), geometric shapes (point sets), as well as sets consisting of the geometric shapes. There are four main types of mappings of interest for school Mathematics:

- 1) *mapping of numerical sets into numerical sets;*
- 2) *mapping of numerical sets into point sets;*
- 3) *mapping of sets of geometric shapes into numerical sets;*

4) *mapping of point sets into point sets.*

The most important are the numerical functions of a numerical argument (type 1 mappings). They are the main object of study in the Mathematical Analysis.

The *coordinate method* establishes a connection between the numerical and geometric sets. For each $x \in R$ corresponds a point $M(x)$ on the coordinate line, and this mapping is bijective. For each triple of coordinates corresponds a point $M(x, y, z)$ in the coordinate space.

Similarly, a connection between pairs of numbers and points of the coordinate plane is established. These correspondences can be considered as a mapping from R^k to Π_k , and as a mapping from Π_k to R^k , $k=1,2,3$. Mapping from R^k to a particular set of geometric figures allows us to determine the number of parameters on which the elements of this set depend.

For example, the set of circles on a plane is a three-parameter family, since any circle is uniquely defined by a triple of numbers (a,b,r) , where a and b are the coordinates of its center, and r is the length of the radius.

The measurement of geometric quantities is an example of the mapping of the sets of geometric figures into the numerical sets. In the process of measurement, for each geometric figure that belongs to some set of figures is matched the non-negative number – the measure of this figure, and the conditions of additivity and invariance must take place.

The geometric transformations are the bijective mappings Π_k on Π_k , $k=1,2,3$. Each such transformation can be specified by a set of the numerical functions. For example, a parallel transfer transforms a point $M(x, y)$ into a point $M'(x', y')$, where $x' = x + a$, $y' = y + b$.

Despite the considered types of mappings, the other mappings are also considered in school Mathematics: expressions into functions, functions into functions, functions into numbers, etc. For example, the differentiation matches each differentiated function with its derivative, that is, it is a mapping of a set of differentiated functions into a set of functions. Calculation of the definite integral of a function is a mapping of a set of integral functions into a set of numbers, etc.

3.4. Operations on the mappings in the school Mathematics

In the school curriculum, the mappings of numerical sets are usually considered, where for each element $x \in X$ corresponds a unique element $y \in Y$. Here are the main operations of mappings studied in school [6], [8]:

1. Arithmetic operations:

If we have two functions $f(x)$ and $g(x)$ with the same domain, then we can create new functions:

Sum and difference: $(f+g)(x) = f(x) + g(x)$; $(f-g)(x) = f(x) - g(x)$;

Product: $(f \cdot g)(x) = f(x) \cdot g(x)$;

Quotient: $(f/g)(x) = (f(x))/(g(x))$, if $g(x) \neq 0$.

For example, if $f(x) = x$, and $g(x) = \sin x$, then their sum will be the new function $h(x) = x + \sin x$.

Let us consider constructing the graph of the function $y = f_1(x) + f_2(x)$, if the graphs $y = f_1(x)$ and $y = f_2(x)$ are given.

Recall that the *graph* of a function is the set of all points $y = f(x)$ in the plane, whose coordinates x, y satisfy the relation $y = f(x)$.

Note that the domain of a function $y = f_1(x) + f_2(x)$ is the set of those values x_0 , for which both functions $y = f_1(x)$ and $y = f_2(x)$ are defined, i.e. the domain is the intersection of the domains of definitions of the functions $y = f_1(x)$ and $y = f_2(x)$.

Let the point (x_0, y_1) belongs to the graph $y = f_1(x)$, that is $y_1 = f_1(x_0)$, and the point (x_0, y_2) belongs to the graph $y = f_2(x)$, that is $y_2 = f_2(x_0)$.

Then the point $(x_0, y_1 + y_2)$ belongs to the graph of the function $y = f_1(x) + f_2(x)$ (since $f_1(x_0) + f_2(x_0) = y_1 + y_2$), and any point on this graph can be obtained in this way. Therefore, the graph of the function $y = f_1(x) + f_2(x)$ can be obtained from the graph $y = f_1(x)$ and $y = f_2(x)$ by replacing each point (x_0, y_1) of the graph of the function $y = f_1(x)$ with a point $(x_0, y_1 + y_2)$, where $y_2 = f_2(x_0)$, that

is, by shifting each point (x_0, y_1) of the graph of the function $y = f_1(x)$ along the axis y by the value $y_2 = f_2(x_0)$. In this case, only such points x_0 are considered for which both functions $y = f_1(x)$ and $y = f_2(x)$ are defined. This method of construction is called *adding the graphs* of the functions $y = f_1(x)$ and $y = f_2(x)$.

Example 3.1. In Fig. 3, the graph of the function $y = x + \sin x$ is constructed using the method of adding graphs [13]:

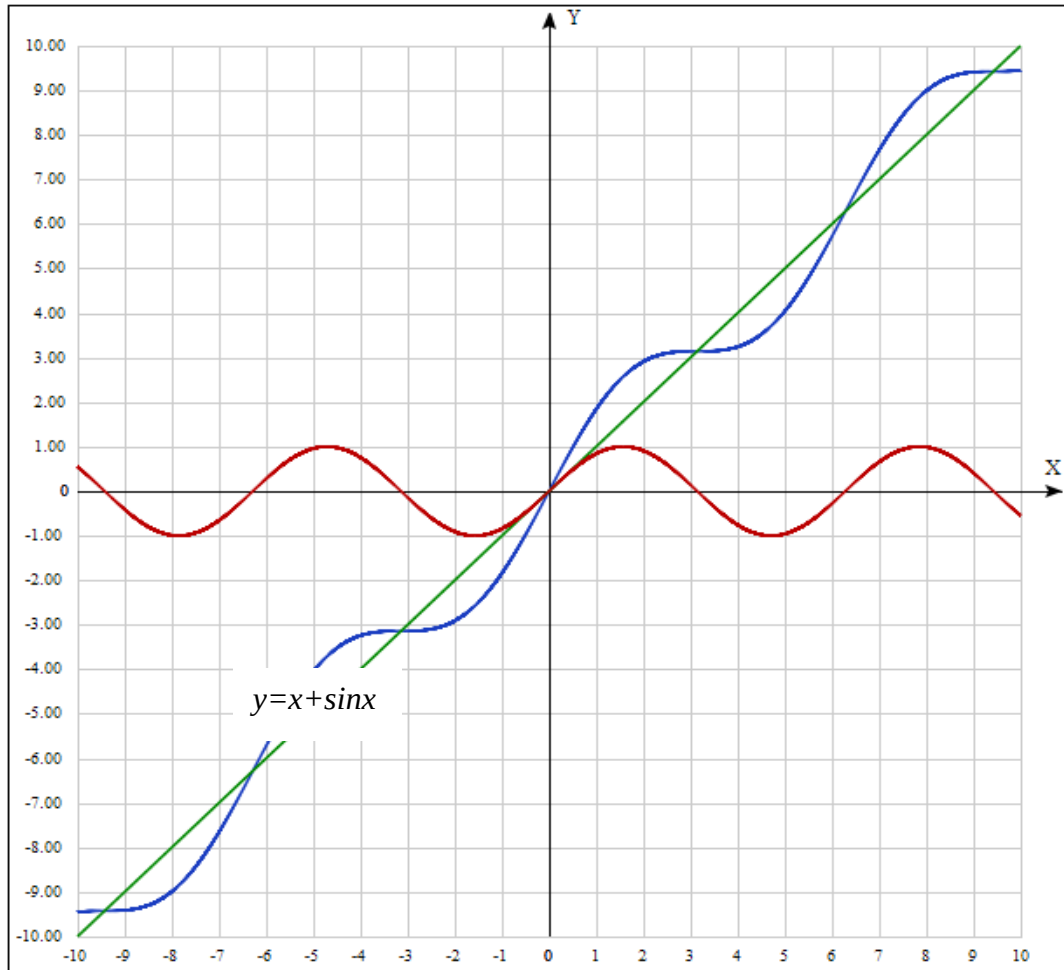


Fig. 3

2. Composition of mappings (compound function)

Let f and g be two functions. Substituting the function $x = g(t)$, instead of the argument x , under the function sign $y = f(x)$, we obtain a new function $y = f(g(t))$, in which the argument is t . The function $y = f(g(t))$ is called the *composition* of the function f and g .

Example 3.2. Let $f(x) = \frac{x^2}{x+1}$. Find $f\left(\frac{t}{t^2+1}\right)$.

So,

$$f\left(\frac{t}{t^2+1}\right) = \frac{\left(\frac{t}{t^2+1}\right)^2}{\left(\frac{t}{t^2+1}\right)+1} = \frac{\frac{t^2}{(t^2+1)^2}}{\frac{t+t^2+1}{t^2+1}} = \frac{t^2}{(t^2+t+1)(t^2+1)}, \text{ since } t^2+1 \neq 0.$$

3. Inverse mapping

Let $f(x)$ be a function defined on the interval $[a, b]$. If this function is not monotonic, then it may happen that two different values of the argument correspond to the same value of the function (Fig. 4):

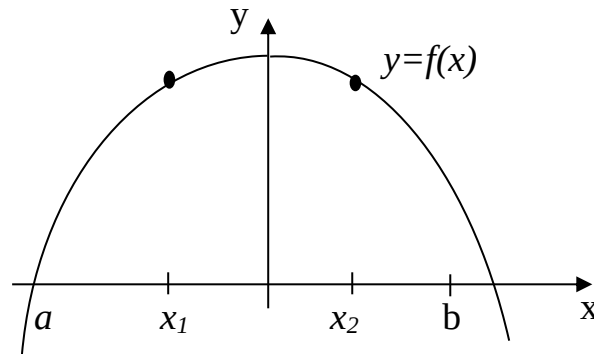


Fig. 4

Here, for the argument value, the corresponding function values can be uniquely determined. However, the inverse transition (from y to x) here will not be unique.

For example, the function takes values y_0 at two points x_1 and x_2 .

In order for the inverse transition to be unique, we are limited by considering the monotone functions.

If we agree to denote the inverse function by g , then its definition can be formulated as follows: equality $x_0 = g(y_0)$ by definition means that at the point x_0 the original function takes the value y_0 , that is, $y_0 = f(x_0)$.

1. To find the inverse of a monotonic function $y = f(x)$, you need to replace the letters x and y , $x = f(y)$ and from the resulting equality, as from the equation, find y .

For example, for the function $f(x)=2x+3$, the inverse will be the function $f^{-1}(x)=\frac{x-3}{2}$.

2. If $f(x)$ is a monotonic function, and $g(x)$ is inverse to it, then the function $g(x)$ is also monotonic, if f is increasing, then g is increasing, if f is decreasing, then g is decreasing.

3. When moving from a function to its inverse function, the domain and set of values are interchanged.

4. The graphs of the inverse functions $f(x)$ and $g(x)$ are symmetric to each other with respect to the line $y = x$.

5. For any x , belonging to the domain of the function $f(x)$, the relation $g(f(x)) = x$ is hold; for any x , belonging to the domain of the inverse function $g(x)$, the relation $f(g(x)) = x$ is hold.

4. Geometric transformations of graphs.

When studying the numerical functions, the graph transformations are considered. This operation is associated with the composition of arbitrary numerical functions and some standard functions such as, $x \rightarrow x + a$, $x \rightarrow ax$, $x \rightarrow |x|$.

Depending on the order in which the composition of functions occurs, we obtain from the function f new functions whose values at a given value of x are equal to $f(x + a)$, $f(ax)$, $f|x|$, $f|x + a|$, $f(x) + a$, $a f(x)$, $|f(x)|$, $|f(x + a)|$ etc. The compositions are also performed, as a result of which we obtain the functions $f(x + a) + b$, $b f(x + a)$, $|f(x) + a|$, $bf(|x| + a)$ etc. From the school course of Mathematics, we know how to construct the graphs of such functions based on a given graph of the function f .

3.5. Continuous and homeomorphic mappings in school Mathematics

A mapping f of a topological space X into a topological space Y is called *continuous* if the inverse image of an arbitrary open set A of Y is open in X . An open set is an abstract concept that generalizes the idea of an open interval on the real number line. If the topology from X to Y is given by a metric, then any isometric mapping $f: X \rightarrow Y$ is homeomorphic.

In particular, any movement of a plane or space is homeomorphic. All affine transformations of a plane and space, in particular, similarity transformations, are homeomorphisms, that is, in the general case, homeomorphisms can change angles, lengths, areas, volumes, and curvature, stretch objects, twist, crumple and bend. For example, any segment $[a; b]$ is homeomorphic to the segment $[0; 1]$; a circle is homeomorphic to an ellipse or a square (they can be stretched into each other), but not homeomorphic to a segment.

The properties of a space that are preserved under homeomorphisms are called *topological*. The most important of these for the school Mathematics course are the properties of connectedness, compactness and dimensionality. Recall that a connected set is a topological space or subset that cannot be represented as the union of two non-empty separated (open or closed) sets. In other words, it is a set consisting of one "whole" part, in which any two points can be connected by a continuous line (path) that belongs entirely to this set. The figures studied in school Mathematics usually consist of a finite number of connected components. For example, the interval $(0; 1)$ on the number line, a circle, a sphere, a line, a plane are connected sets. *The continuous image of a connected set is connected.*

Many figures considered in school Mathematics are closed and bounded (circle, circle, polygons, etc.). Closed and bounded figures on the plane and in space are called *compact*. The image of a closed bounded figure under continuous mapping is closed and bounded.

The dimension of any line studied in a school Mathematics course (circles, hyperbolas, parabolas, function graphs, etc.) is 1, the dimension of any surface is 2, and the dimension of a sphere and other spatial bodies is 3.

Control questions

1. Formulate the definition of a mapping from the set X to the set Y and a mapping of the set X to the set Y .

2. What mappings are called the injective, surjective, and bijective? Give the examples of them in the school course of Mathematics.
3. What is a mathematical structure?
4. Formulate the definition of a morphism of a structure.
5. What are the main types of mappings of interest to school Mathematics? Give the examples of each of these mappings.
6. What operations on mappings are performed in the school course of Mathematics?
7. Name the continuous and homeomorphic mappings studied in the school Mathematics course.

§ 4. Functions in the school course of Mathematics

4.1. Methods of specifying the function

In addition to the known methods of specifying a function (analytical, graphical, tabular, composition of functions, parametric specification of a function), the other specifications of the functions are also possible.

For example, the exponential function $y = e^x$ can be specified in the following way:

$$e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n,$$
$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

The following representation of the complex exponent is also known:

$$e^{ix} = \cos x + i \sin x.$$

For sufficiently small neighborhoods of almost all argument values, analytic functions can be represented as the sum of power series:

$$f(x) = a_0 + a_1(x - x_0) + \dots + a_n(x - x_0)^n,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

For a geometric progression whose denominator $|x| < 1$, we have:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots.$$

Also the integral specification of a function is known, for example:

$$\ln x = \int_1^x \frac{dx}{x}, x > 0.$$

4.2. Continuous functions in the school Mathematics

Continuity of a function is one of those concepts that is often explained intuitively in school (a function is continuous if its graph can be drawn without lifting the pencil from the paper), but it is the foundation for a course in Mathematical Analysis (derivatives and integrals).

In the school course, a function $f(x)$ is called continuous at the point x_0 , if three conditions are hold:

- 1) the function is **defined** at this point (i. e. $f(x_0)$ exists);
- 2) there exists a limit of the function at this point: $\lim_{x \rightarrow x_0} f(x)$;
- 3) the limit of the function at this point is equal to the value of the function at this point: $\lim_{x \rightarrow x_0} f(x) = f(x_0)$.

If a function is continuous at every point on an interval, it is called continuous on that interval.

Most functions studied in school are continuous in their domain:

- 1) *a linear function* is always continuous;
- 2) *a quadratic function* is always continuous;
- 3) *power, exponential, logarithmic, and trigonometric functions* are continuous at all points where they are defined.

If at least one of the continuity conditions is violated, then the function has a discontinuity. For example:

- 1) $y = \frac{1}{x}$ – at the point $x=0$ the function has a discontinuity of the second kind (one-sided limits are different and equal to $+\infty$ and $-\infty$);
- 2) $y = \frac{x^2-4}{x-2}$ – at the point $x=2$ the function is undefined, but its graph looks like a straight line with a "punctured" point (removable gap);

3) $f(x) = \begin{cases} x^2, & x \leq 1 \\ x + 5, & x > 1 \end{cases}$ – at the point $x=1$ the function has a "jump" type discontinuity (one-sided limits are different and equal to 1 and 6, a "jump" is formed between parts of the graph).

In school problems, points that are “suspicious” for a discontinuity are points where:

- 1) the denominator is 0;
- 2) the formula changes (in piecewise functions);
- 3) the argument of the logarithm function or, for example, the tangent, goes beyond the limits of permissible values.

Discontinuous functions in school Mathematics also include the functions $y=[x]$ (the integer part of a number, the function has a discontinuity at each integer point $x=1,2,3,\dots$) and $y=\{x\}$ (the fractional part of a number $\{x\}=x-[x]$, the function also has a discontinuity at each integer point $x=1,2,3,\dots$).

4.3. Axiomatic definitions of linear, exponential, logarithmic, power functions, trigonometric functions of a numerical argument and their properties. The theorems of existence and uniqueness of these functions

Definition 4.1. A linear function is any continuous homomorphism of the group $\mathbf{R}^+$ into itself, that is, any function f from the set $\mathbf{R}$ into itself that has the properties::

- 1) $f(x + y) = f(x) + f(y)$;
- 2) f is continuous.

Definition 4.1*. A linear function with coefficient a , $a \in \mathbf{R}$, is any linear function that has the additional property:

- 3) $f(1) = a$.

Let us list the properties of a linear function (continue the numbering):

- 4) $f(0) = 0$;
- 5) $f(-x) = -f(x)$;
- 6) $f(cx) = cf(x)$ for any $c, x \in \mathbf{R}$.

Theorem 4.1. Let $a \in R$. Then exists the unique linear function $l_a(x)$ with the coefficient a , $l_a(x) = ax$ for any $x \in R$.

In school Mathematics, the function $y = kx + b$ is called a linear one. In the Higher Mathematics, it is called an affine-linear or affine function. In any case, the $y = kx + b$ for $b \neq 0$ does not satisfy the definition 4.1.

Definition 4.2. An exponential function is any continuous homomorphism of the group R^+ into the group R , that is, any function $f(\cdot)$ from the set R into the set $R > 0$, which has the properties:

- 1) $f(x + y) = f(x) \cdot f(y)$;
- 2) f is continuous.

Definition 4.2*. An exponential function with base a , where $a > 0$, is an exponential function that has the additional property $f(1) = a$.

Any exponential function we denote by $f(x) = a^x$. Let us present the properties of the exponential function:

- 1) $a^{x+y} = a^x \cdot a^y$;
- 2) a^x is continuous;
- 3) $a^1 = a$;
- 4) $a^0 = 1$;
- 5) $a^{-x} = (a^x)^{-1}$;
- 6) $(a^b)^x = a^{bx}$ for any $b, x \in R$;
- 7) the function $f(x) = a^x$ is monotonically increasing if $a > 1$ and monotonically decreasing if $0 < a < 1$.

Theorem 4.2. There exists the unique exponential function with the base a , $a > 0$, i. e. the unique function f from R into $R_{>0}$ with the properties:

- 1) $f(x + y) = f(x) \cdot f(y)$ for any $x, y \in R$;
- 2) f is continuous;
- 3) $f(1) = a$.

Definition 4.3. A logarithmic function is any continuous homomorphism of the group R into the group R^+ , that is, any function from the set $R > 0$ into the set R that has the properties:

$$1) f(x \cdot y) = f(x + y);$$

2) f is continuous.

Definition 4.3*. A logarithmic function with the base a , where $a > 0$, is the logarithmic function, which has the properties:

$$3) f(a) = 1.$$

$$4) f(1) = 0.$$

For $a=1$ the properties 4 and 3 contradict each other. Therefore, a logarithmic function with base $a=1$ does not exist. Often, the restriction $a \neq 1$ is included in the definition of a logarithmic function with base a .

Denote by $f(x) = \log_a x$. Let us present the properties of the logarithmic function:

$$1) \log_a(x \cdot y) = \log_a x + \log_a y, \quad \forall x, y \in R_{>0};$$

$$2) f(x) = \log_a x \text{ is continuous};$$

$$3) \log_a a = 1;$$

$$4) \log_a 1 = 0;$$

$$5) \log_a x^{-1} = -\log_a x \text{ if } x > 0;$$

$$6) \log_a \left(\frac{x}{y} \right) = \log_a x - \log_a y, \quad \forall x, y \in R_{>0};$$

$$7) \log_a(a^x) = x, \quad \forall x \in R;$$

8) the logarithmic function with base a is a bijection that is the reciprocal of the exponential function with base a ;

9) the logarithmic function is monotonically increasing if $a > 1$ and monotonically decreasing if $0 < a < 1$;

$$10) \log_a(x^b) = b \log_a x, \quad \forall b \in R, x > 0;$$

$$11) \log_a x = \log_a c \cdot \log_c x.$$

Theorem 4.3. There exists the unique logarithmic function with the base $a > 0$, i. e. the function from $R_{>0}$ into R with the properties:

$$1) f(x \cdot y) = f(x + y);$$

2) f is continuous;

$$3) f(a) = 1.$$

Definition 4.4. A power function is any continuous homomorphism of a group R into itself, that is, any function that maps the set $R_{>0}$ into itself, which has the properties:

- 1) $f(x \cdot y) = f(x) \cdot f(y) \quad \forall x, y \in R_{>0}$;
- 2) f is continuous.

Definition 4.4*. A power function with exponent a , where $a \in R$, is a power function, that has the additional property:

- 3) $f(2) = 2^a$.

In the property 3, the number 2 does not play any special role. Any positive number can be written except 1. The notation 2^a is understood to mean the value of an exponential function with base 2.

For some values of a , the power function x^a allows the extension to a wider domain than $R_{>0}$. For example, if $a = n \in N$, then by evenness or oddness (depending on n) x^a is extended to $R_{<0}$. In addition, by continuity 0^n is equal to 0. If $a = -n$, $n \in N$, then x^a is extended only to $R_{<0}$. If $a > 0$, then $\lim_{a \rightarrow 0+} x^a = 0$. In order not to violate the continuity of the function x^a , in this case we set $0^a = 0$.

For odd $n \in N$, the function $x^{1/n}$ admits a unique extension to the entire number line, while for even $n \in N$ this extension is impossible. This situation is related to the injectivity or non-injectivity of the function x^n , since the equality $\left(x^{\frac{1}{n}}\right)^n = x$ essentially defines $x^{\frac{1}{n}}$ as a function inverse to x^n , so the function $x^{\frac{20}{13}}$ can be considered defined for $\forall x \in R$, and the function $x^{\frac{13}{20}}$ only for non-negative x .

Theorem 4.4. There exists the unique power function with the base a , where $a \in R$.

The properties of the power functions:

- 4) $1^a = 1$;
- 5) $\left(\frac{1}{x}\right)^a = \frac{1}{x^a}$;
- 6) $(x^a)^b = x^{ab}$;
- 7) $x^0 = 1$;

- 8) the function x^a is monotonically increasing if $a > 0$ and monotonically decreasing if $a < 0$.

The axiomatic definition of trigonometric functions is based on the following properties:

- 1) the axioms of addition (for example, $\sin(a+b)=\sin a \cdot \cos b + \cos a \cdot \sin b$, $\cos(a-b)=\cos a \cdot \cos b + \sin a \cdot \sin b$);
- 2) the value of functions at certain angles (например, $\cos 0=1$, $\sin 0=0$);
- 3) the fundamental relationship $\sin^2 x + \cos^2 x = 1 \quad \forall x \in R$;
- 4) periodicity and symmetry (for example, $\sin(x+2\pi)=\sin x$ etc. and properties of evenness and oddness $\cos(-x)=\cos x$, $\sin(-x)=-\sin x$).

There are also the following approaches to the axiomatic definition:

- 1) complex exponent $e^{ix}=\cos x + i\sin x$;
- 2) cosine and sine are defined as solutions of differential equations, for example, $y'' = -y$.

Control questions

1. What are the ways of specifying a function?
2. How can a function be specified using power series?
3. What is an integral representation of a function?
4. Give the examples of continuous functions and functions with discontinuities that are found in the school course of Mathematics.
5. Formulate the axiomatic definitions of linear, exponential, logarithmic, power functions, trigonometric functions of a numerical argument.
6. Describe the properties of the functions listed above.
7. Formulate the theorems of existence and uniqueness of linear, exponential, logarithmic and power functions.

§ 5. Polynomials, equations and inequalities

5.1. Actions on the polynomials

In the school curriculum, the study of polynomials begins in the 7th grade in Algebra lessons. This topic includes the definition of a polynomial and its standard form, addition and subtraction of polynomials, multiplication of a monomial by a polynomial and a polynomial by a polynomial, formulas for abbreviated multiplication (square of the sum, difference of squares, etc.). In high school (especially in classes with in-depth study), the topics are actually part of higher Algebra. Namely, division of polynomials by a "corner", Bezu's theorem, Horner's scheme, Vieta's theorem for equations of higher degrees, integer roots of a polynomial, linear factorization.

A *polynomial* of x is an expression of the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-2}x^2 + a_{n-1}x + a_n, \quad (5.1)$$

where $a_0, a_1, \dots, a_n$ are some numbers called the coefficients of the polynomial, x is a symbol that can be substituted with any numerical value [1], [6].

Often the functional symbols are used to denote the polynomials:

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-2}x^2 + a_{n-1}x + a_n.$$

Then the value of this polynomial at $x=c$ is denoted by $f(c)$:

$$f(c) = a_0c^n + a_1c^{n-1} + a_2c^{n-2} + \dots + a_{n-2}c^2 + a_{n-1}c + a_n.$$

The coefficient in expression (5.1) is called the *free term*. It is obvious that when $x=0$ we obtain $f(0)=a_n$, and when $x=1$ we have that

$$f(1) = a_0 + a_1 + a_2 + \dots + a_{n-1} + a_n.$$

Therefore, the value of an arbitrary polynomial when $x=1$ is equal to the sum of all coefficients of this polynomial. These two simple remarks are often used in solving problems.

The identical transformations of polynomials are opening the brackets, reducing like terms, and permuting terms.

If $a_0 \neq 0$, then x^n is called the leading term, a_0 is the leading coefficient, and the polynomial is called a polynomial of the n -th degree. The notation (5.1) is the canonical notation of a polynomial of the n -th degree.

Example 5.1. Find the sum of the coefficients of the polynomial which we'll obtain from the polynomial

$$1 + (x^2 - 6x + 5)(x^5 + 3x^4 - 2x^3 + x^2 - x)^3 + (x^2 - 3x + 1)^{25}(x^3 + 5x + 7)$$

by reducing it to the canonical form.

Solution. We have that $f(1) = 1 + (-1)^{25} \cdot 13 = -12$. Therefore, the sum is -12 .

Let us have two polynomials $P(x)$ and $Q(x)$:

$$P(x) = a_0x^n + a_1x^{n-1} + \dots + a_n,$$

$$Q(x) = b_0x^m + b_1x^{m-1} + \dots + b_n.$$

The equality $P(x) = Q(x)$ is true only if $m = n$, $a_0 = b_0, \dots, a_n = b_n$.

The following theorems are hold:

Theorem 5.1. If $P(x) = Q(x)$, then for any number c the equality $P(c) = Q(c)$ holds.

Theorem 5.2. If for any number c the equality $P(c) = Q(c)$ is hold, then $P(x) = Q(x)$.

Theorem 5.3. If $P(x) \neq Q(x)$, then there exists a number c such that $P(c) \neq Q(c)$.

Theorem 5.4. If there exists such a number c such that $P(c) \neq Q(c)$, then $P(x) \neq Q(x)$.

Theorem 5.1 (and hence, the equivalent Theorem 5.4) is obvious. On the contrary, Theorem 5.2 (and the equivalent Theorem 5.3) is not at all obvious.

Example 5.2. A student multiplied the polynomials and obtained the following result: $(x^5 - x^4 + x^2 - x + 1)(x + 1) = x^6 + 1$. Check whether the multiplication was performed correctly.

Solution: Let $P(x) = (x^5 - x^4 + x^2 - x + 1)(x + 1)$ and $Q(x) = x^6 + 1$.

If $x=0$, then we have $P(0)=1$, $Q(0)=1$, then $P(0) = Q(0)$. Similarly, $P(1)=2$, $Q(1)=2$, i.e. $P(1)=Q(1)$.

Can we say that $P(x)=Q(x)$, i.e., that the multiplication is done correctly? No theorem gives the answer.

Further, for $x = -1$, $P(-1)=0$, $Q(-1)=2$, i.e., $P(-1) \neq Q(-1)$.

Therefore, the multiplication is not done correctly.

Theorem 5.5. *Let $P(x)$ and $Q(x)$ be two polynomials, each of degree not exceeding n . Then, if the values of the polynomials coincide at $n+1$ points, then $P(x)=Q(x)$.*

If $P(c_1) = Q(c_1)$, $P(c_2) = Q(c_2)$, ..., $P(c_{n+1}) = Q(c_{n+1})$ (where $c_1, c_2, \dots, c_{n+1}$ are different numbers), then $P(x)=Q(x)$.

Theorem 5.2 follows from the Theorem 5.5.

The following theorems are often useful in solving a number of problems.

Theorem 5.6. *Let $P(x)$ and $Q(x)$ be two nonzero polynomials. Then the degree of the polynomial $P(x) \cdot Q(x)$ is equal to the sum of the degrees of the polynomials $P(x)$ and $Q(x)$, and the highest coefficient is equal to the product of the highest coefficients of the polynomials $P(x)$ and $Q(x)$.*

Theorem 5.7. *The degree of the polynomial $P(x)+Q(x)$ (or $P(x)-Q(x)$) does not exceed the largest of the degrees of the polynomials $P(x)$ and $Q(x)$.*

Let's consider the dividing of the polynomials. Let $f(x)$ and $g(x)$ be two polynomials, $g(x) \neq 0$. If there exists a polynomial $q(x)$ such that $f(x) = g(x) \cdot q(x)$, then we say that the polynomial $f(x)$ is divisible by $g(x)$ ($g(x)$ is the divisor, $q(x)$ is the quotient).

The simple examples show that one polynomial is not always divisible by another. For example, a polynomial $x^2 + 1$ is not divisible by a $(x-1)$ whole. If $x^2 + 1$ was divisible by $(x-1)$, then $x^2 + 1 = (x-1) \cdot q(x)$, but for $x=1$, $x^2 + 1 = 2$ and $(x-1) \cdot q(x) = 0$. Therefore, the written equality has no meaning for any $q(x)$.

So, in the set of polynomials, the division operation cannot always be performed. However, there is a more general operation: division with a remainder (if the divisor is not equal to 0).

Let $f(x)$ and $g(x)$ be two arbitrary polynomials, and $g(x) \neq 0$. Then there exist the polynomials $q(x)$ and $r(x)$, such that the equality $f(x) = g(x) \cdot q(x) + r(x)$ is hold, where $r(x)$ either is equal to 0 or has a lower degree than the polynomial $g(x)$. By these conditions the polynomials $q(x)$ and $r(x)$ are defined uniquely.

5.2 Algebraic equation and its roots. Bezout's theorem. The fundamental theorem of Algebra

The number c is called a root of the polynomial $f(x)$ if $f(c)=0$. The problem of “finding all the roots of a given polynomial $f(x)$ ” is also usually formulated as follows: solve the equation $f(x)=0$.

In other words, if we are talking about the equation $f(x)=0$, then we do not mean that the notation $f(x)=0$ means the equality of the polynomials (that is, that all coefficients are equal to 0).

Note that the roots of the polynomial $f(x)$ are also called the roots of the equation $f(x)=0$. We emphasize that solving an equation means finding all of its roots [4], [6].

There are various equations in the school course of Mathematics: irrational, logarithmic, exponential, trigonometric, and others. An *algebraic equation* is an equation of the form $f(x)=0$, where $f(x)$ is a polynomial. If $f(x)$ is a polynomial of the n -th degree, then the equation $f(x)=0$ is called an *algebraic equation of the n -th degree*.

When solving the algebraic equations, the following theorem (Bezout's theorem) is useful.

Theorem 5.8. *The remainder of the division of a polynomial $f(x)$ by $(x - a)$ is equal to $f(a)$.*

Indeed, $f(x) = (x - a) \cdot q(x) + r(x)$, $r(x) \neq 0$, the degree of the remainder $r(x)$ is less than the degree of the divisor $(x - a)$ or is equal to 0. Therefore, $r(x)=r$ is a number,

$$f(x) = (x - a) \cdot q(x) + r, \quad f(a) = 0 + r, \quad f(a) = r.$$

Theorem 5.9. *If a is a root of a polynomial $f(x)$, then this polynomial is divisible by $(x - a)$.*

If we know the root $x=a$ of the algebraic equation $f(x)=0$, then finding the remaining roots of this equation reduces to solving the equation $g(x)=0$, which has a degree 1 less than the original equation.

Theorem 5.10. *If all coefficients of the polynomial*

$$f(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$$

are integers, then every integer root of this polynomial is a divisor of the free term a_n .

Theorem 5.11. *An arbitrary polynomial of degree n*

$$f(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$$

can be represented as the product

$$f(x) = a_0(x - c_1)(x - c_2) \dots (x - c_n).$$

This decomposition is unique up to the order of the multipliers. The proof of Theorem 5.11 is based on the following theorem.

Theorem 5.12. *An arbitrary polynomial $f(x)$ whose degree is different from 0 has at least one root.*

The proof of this theorem is not elementary, according to the methods of proof it is not a non-algebraic theorem.

Theorem 5.13. *Let $f(x) = a_0(x - c_1)(x - c_2) \dots (x - c_n)$ be the decomposition of a polynomial of degree n into multipliers of degree 1. Then the numbers $c_1, c_2, \dots, c_n$ are the roots of the polynomial $f(x)$ and this polynomial has no other roots.*

If among the numbers $c_1, c_2, \dots, c_n$ there are those that coincide, then we say that the polynomial has multiple roots. If among the numbers the number c occurs k times, and all the others are different from c , then c is called a k -fold root of the polynomial $f(x)$.

Theorem 5.14 (refinement of the fundamental theorem of algebra). *Every algebraic equation of the n th degree has exactly n roots.*

Example 5.3. Find the remainder from dividing a polynomial

$$x^{99} + x^3 + 10x + 5$$

by a polynomial $x^2 + 1$.

Solution. By the definition of division, we have the identity

$$x^{99} + x^3 + 10x + 5 = Q(x)(x^2 + 1) + ax + b,$$

which is valid everywhere in the domain of complex numbers. As x , we need to choose one of the roots of the expression $x^2 + 1$, for example, $x=i$. Substitute $x=i$ and get

$i^{99} + i^3 + 10i + 5 = ai + b$, that is, $8i + 5 = ai + b$, from where $a = 8, b = 5$. Therefore, the remainder is $8x + 5$.

5.3. Different approaches to defining the concept of an equation (inequality)

In modern Mathematics, there are three main approaches to the interpretation of equations and inequalities:

- 1) **expression approach (classical)**. An equation is considered as an equality of two mathematical expressions, that is, it is consisting of two expressions connected by the "=" sign (or the ">" and "<" signs for inequalities). The main attention is paid to formal transformations of expressions (simplification, opening of brackets) to find the value of the variable;
- 2) **functional approach**. The equation is considered as a question of common points of the values of two functions. The equation $f(x) = g(x)$ is the problem of finding the values of the argument x at which the values of the functions f and g coincide. Solving the equation means finding the abscissas of the intersection points of the graphs $y = f(x)$ and $y = g(x)$. The inequality $f(x) > g(x)$ determines the intervals where the graph of one function lies above the other;
- 3) **logical-semantic approach (through the predicates)**. This is the most rigorous approach that uses the concepts of mathematical logic. *An equation (or inequality)* is a *predicate* (a sentence with a variable), which is transformed into a true or false statement when a specific value of the variable is substituted from a certain set. That is, the variable x is an object from a certain domain of definition. Solving an equation means finding the truth set of this predicate (all values of x for which the statement becomes true). If the truth set is empty, the equation has no solutions.

The equal sign is used in Mathematics very often and the meaning given to this sign is not always the same. For example, (5.2), (5.3), (5.4), where (5.2), (5.3), (5.4) are some expressions:

$$\frac{651}{257} = 2 \frac{137}{257}, \quad (5.2)$$

$$1236 = 2^2 * 3 * 103, \quad (5.3)$$

$$\log_{\sqrt{2}} \left(\frac{1}{2} \right)^5 = 10. \quad (5.4)$$

The meaning of "=" in (3.2), (3.3), (3.4): the truth of such a statement means that the same number is on the right and on the left sides. We will call such statements the numerical equalities.

In another sense, the sign "=" is used when we are talking about the equality of functions. Two functions $f(x)$ and $g(x)$ are equal (i.e. coincide) if, firstly, the domains of definition of these functions coincide, and, secondly, for $\forall x_0 \in D(f)$: $f(x_0)=g(x_0)$.

Often in such cases the sign " $\equiv$ " is used instead of "=". The notation $f(x)=g(x)$ or $f(x)\equiv g(x)$ is also called *an identity*.

For example, $(x + 1)^2 = x^2 + 2x + 1$, $\log_2 2^x = x$.

Sometimes when considering identities it is necessary to limit the domain of definition of functions. Namely, we will say that the equality $f(x)=g(x)$ is an identity on the set M if, firstly, the set M is contained in the domain of definition of each of the functions $f(x)$, $g(x)$, and, secondly, the numerical equality $f(x_0)=g(x_0)$ is valid for $\forall x_0 \in M$.

We write $f(x)\equiv g(x)$ on the set M or $f(x)=g(x)$ for $x \in M$.

Example 5.4. Equality $\sqrt{x^2} = x$ is an identity on the set of non-negative numbers, i.e. $\sqrt{x^2} \equiv x$, when $x \geq 0$. Although the domain of these functions is $(-\infty; +\infty)$, their values coincide only on the set of non-negative numbers. On the set of all real numbers, the relation is not an identity.

In a completely different sense, the sign "=" is used when considering equations.

An equation with one unknown x has the following general form

$$f(x)=g(x), \quad (5.5)$$

where $f(x)$ and $g(x)$ are arbitrary functions. If we say that (5.5) is an equation, this means that the equality (4.5) is considered as an indefinite statement (for some values of x it is true, for others it is false), that is, we are interested in finding the roots.

The number a is called a root of equation (4.5) if, first, this number belongs to the domain of both the functions $f(x)$ and $g(x)$, and, secondly, $f(a)=g(a)$.

Solving an equation means finding all its roots (or proving that the equation has no roots).

5.4. Relation of implication and equivalence on the set of equations (inequalities). Basic transformations of equations (inequalities) from the point of view of their equivalence

In the process of solving equations, we usually perform some transformations.

There is a basic rule: you cannot perform transformations that can lead to the loss of roots. Sometimes, with some transformations, new roots may appear. This is why it is necessary to check the roots.

Let $f(x)=g(x)$ be a given equation, and $f_1(x)=g_1(x)$ be some new equation formed as a result of the transformations. We will say that when going from the equation $f(x)=g(x)$ to $f_1(x)=g_1(x)$ there is a loss of roots if there exists a number x_0 (at least one) that is a root of the equation $f(x)=g(x)$ and is not a root of the equation $f_1(x)=g_1(x)$. Further, a number x_0 is called a foreign root if this number is a root of the equation $f_1(x)=g_1(x)$ and is not a root of the equation $f(x)=g(x)$.

Example 4.5. Solving the equation $x^3 = x$, by dividing both its parts by x , we get

$$x^2 = 1; x_{1,2} = \pm 1.$$

But $x=0$ is also a root of the equation. Therefore, by reducing by x , we have lost one root.

Definition 5.1. Two equations $f(x)=g(x)$ (5.6) and $f_1(x)=g_1(x)$ (5.7) are called equivalent on some set M if they have the same solutions on this set, i.e. each root of equation (5.6) belonging to the set M is a root of equation (5.7), and vice versa, each root of equation (5.7) belonging to the set M is a root of equation (5.6).

Example 5.6. The equations $x = 0$ and $x(x^2 + 1) = 0$ are equivalent on the set $\mathbf{R}$ ($x=0$ is the unique root) and are not equivalent on the set $\mathbf{C}$, where the second equation has two more roots ($x_2 = i, x_3 = -i$).

Definition 5.2. Let some equation $f(x)=g(x)$ be given. The equation $f_1(x)=g_1(x)$ is called a consequence of the equation if no roots are lost when going from the first to the second equations, i.e. all roots of $f(x)=g(x)$ are roots of $f_1(x)=g_1(x)$.

Definition 5.3. We will say that equation $f(x)=g(x)$ (5.6) is equivalent to the disjunction of equations $f_1(x)=g_1(x), f_2(x)=g_2(x), \dots, f_n(x)=g_n(x)$ (5.7) if the following conditions are hold:

- 1) each root of (5.6) is a root of at least one of the equations (5.7);
- 2) any root of any equation (5.7) is a root of equation (5.6).

That is, we can write

$$f(x)=g(x) \leftrightarrow f_1(x)=g_1(x) \vee f_2(x)=g_2(x) \vee \dots \vee f_n(x)=g_n(x).$$

Now let us consider the inequalities. All real numbers are divided into positive, negative and 0. To indicate that a is positive, we use the notation $a>0$.

Recall that the sum and product of positive numbers are also positive numbers. Further, if a is negative, then $-a$ is positive (and vice versa). Finally, for any positive number a , there is a positive rational number r such that $r<a$. These facts are the basis of the theory of inequalities.

By the definition, the inequality $a>b$ (or $b<a$) only holds if $a-b$ is positive.

Let us present the main properties of inequalities. Let $a, b, c \in \mathbf{R}, n \in \mathbf{N}$.

- 1) if $a > b$ i $b > c$, then $a > c$ (transitivity).

Proof: $a > b$, by the definition $a - b > 0$; $b > c$, $b - c > 0$. Hence,

$$a - c = (a - b) + (b - c) > 0$$

as the sum of positive numbers. Hence, $a > c$.

- 2) if $a > b$, then for any c the inequality holds

$$a + c > b + c.$$

Proof: since $a > b$, then $a - b > 0$. Hence,

$$(a + c) - (b + c) = a - b > 0,$$

i. e. $a + c > b + c$.

3) If $a + b > c$, then $a > c - b$.

The proof follows from property 2).

$$a + b > c \quad | +(-b),$$

$$a + b - b > c - b,$$

$$a > c - b.$$

4) if $a > b$ i $c > d$, then $a + c > b + d$.

Proof: according to the definition of inequality, it is sufficient to show that the difference $(a + c) - (b + d) > 0$. We have

$$(a + c) - (b + d) = (a - b) + (c - d).$$

Since the condition implies that $(a - b) > 0$ and $(c - d) > 0$, then

$$(a + c) - (b + d) > 0.$$

Corollary. From properties 2) and 4) the following rule for subtracting inequalities follows: if $a > b$ and $c > d$, then $a - d > b - c$ (to prove it, it is enough to add to both sides of the inequality $a + c > b + d$ add a number $(-c - d)$).

5) if $a > b$, then when $c > 0$ we have that $ac > bc$, and if $c < 0$ we have $ac < bc$.

Proof: if $a > b$, then $a - b > 0$. That's why the sign of $ac - bc = c(a - b)$ coincide with the sign of a number c . If $c > 0$, then $ac - bc > 0$, $ac > bc$. If $c < 0$, then $ac - bc < 0$, $ac < bc$.

6) if $a > b > 0$ and $c > d > 0$, then $ac > bd$.

Proof: $ac - bd = ac - bc + bc - bd = c(a - b) + b(c - d)$. Since $c > 0, b > 0, a - b > 0, c - d > 0$, then $ac - bd > 0$, i. e. $ac > bd$.

7) if $a > b > 0$ and $c > d > 0$, then $\frac{a}{d} > \frac{b}{c}$.

Proof: we have, that $\frac{a}{d} - \frac{b}{c} = \frac{ac - bd}{cd}$. Since the numerator and denominator of the fraction are positive, then to $\frac{a}{d} - \frac{b}{c} > 0$.

Partial case: if $a = b = 1, c > d > 0$, then $\frac{1}{c} < \frac{1}{d}$. This rule is maintained if $c < 0, d < 0$. Hence, if $c > d$ and $cd > 0$, then $\frac{1}{c} < \frac{1}{d}$. If $c > d$ and $cd < 0$ (i. e. $c > 0, d < 0$), then $\frac{1}{c} > \frac{1}{d}$.

Equivalent transformations of equations

1) addition (subtraction) of the same expression:

$$f(x)=g(x) \Leftrightarrow f(x)+h(x)=g(x)+h(x);$$

2) multiplication (division) by a non-zero expression:

$$f(x)=g(x) \Leftrightarrow f(x) \cdot h(x)=g(x) \cdot h(x), h(x) \neq 0;$$

In this case, the range of permissible values must be taken into account.

4) transfer of members:

$$f(x)=g(x) \Leftrightarrow f(x)-g(x)=0;$$

5) opening brackets, reducing like terms;

6) change of the variable (bijective) – if the change is one-to-one, then the equivalence is preserved.

Non-equivalent transformations of equations

1) multiplication by an expression that can be equal to zero (extra solutions may appear);

2) raising to a power (the sign is lost, therefore extraneous roots appear);

3) extracting the root (part of the solutions is lost);

4) logarithmization, exponentiation.

Equivalent transformations of inequalities

1) addition (subtraction):

$$f(x)>g(x) \Leftrightarrow f(x)+h(x)>g(x)+h(x);$$

2) multiplication (division):

$$f(x)>g(x) \Leftrightarrow f(x) \cdot h(x)>g(x) \cdot h(x), h(x)>0,$$

$$f(x)>g(x) \Leftrightarrow f(x) \cdot h(x)<g(x) \cdot h(x), h(x)<0 \text{ (the sign changes);}$$

3) raising to a power (for an even power – non-equivalent transformation, for an odd power – equivalent);

4) application of monotonic functions (if the function increases, then the sign is preserved, if it decreases, then the sign changes).

5.5. Equations with one and two parameters.

Equations in the domain of complex numbers

Equations with parameters and equations in the domain of complex numbers require the special methods of solution because they depend on additional conditions or an extended number system. A parameter is a constant quantity which value is not specified in the condition, but on which the number or type of roots of the equation depends.

An equation with parameters is an equation of the form $f(x; a_1; a_2; \dots; a_n) = 0$, where x is the unknown, $a_1; a_2; \dots; a_n$ are the variable parameters. Solving an equation with parameters means that for each value of the parameter it is necessary to determine whether the equation has solutions, and if so, to find these solutions, which, as a rule, depend on the parameter. Such equations are solved by the analytical and graphical methods.

Example 5.7. Find all values of the parameter a for which the equation $|x^2 - 8|x| + 7| = a$ has less than 4 roots.

Solution. Let us use the graphical method to solve the given equation with the parameter (Fig. 5). Let us select the complete square and write this equation in the form $|(|x| - 4)^2 - 9| = a$. Let $f(x) = |(|x| - 4)^2 - 9|$, $g(x) = a$.

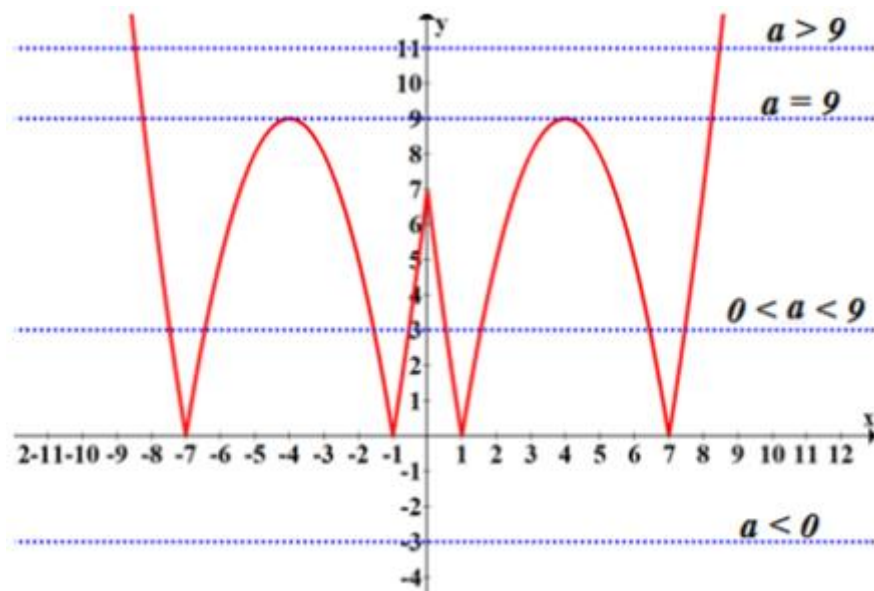


Fig. 5

Let us schematically depict the graphs of the functions $f(x)$ and $g(x)$. The graph of the function $g(x)=a$ is a straight line parallel to the Ox axis. The solutions of the equation are the abscissas of the intersection points of the function graphs.

The straight line will not intersect the graph of $f(x)$ if $a < 0$. If $a > 9$, then the straight line will have less than four intersection points with the graph (namely two). That is, the equation has less than 4 roots if

$$a \in (-\infty; 0) \cup (9; +\infty).$$

An equation with two parameters is called an equation of the form $f(x; a, b)=0$, where a and b are parameters.

Example 5.8. Find all values of the parameters $a, b \in \mathbb{R}$ for which the equation

$$x^2 + ax + b = 0$$

has two different positive roots.

Solution. Let x_1 and x_2 are the roots of the given equation. By Vieta's theorem:

$$x_1 + x_2 = -a, \quad x_1 x_2 = b.$$

In order for the equation to have two distinct roots, it is necessary that

$$D = a^2 - 4b > 0.$$

In order for the roots to be positive, it is necessary and sufficient that

$$\begin{cases} x_1 + x_2 > 0, \\ x_1 x_2 > 0. \end{cases}$$

Combining all the conditions, we obtain the following system for the parameters:

$$\begin{cases} a^2 - 4b > 0, \\ a < 0, \\ b > 0. \end{cases}$$

Therefore, the equation has two distinct positive roots if and only if

$$a < 0, b > 0, a^2 > 4b.$$

An equation in the domain of complex numbers is an equation where the variable z belongs to the set C , $z=x+iy$, where $x, y \in R$, $i^2=-1$.

Solving of the equations:

- 1) 1) *quadratic equations*: with a negative discriminant, the solution is two conjugate complex numbers $z = \frac{-b \pm i\sqrt{|D|}}{2a}$;
- 2) *n-th root*: the equation $z^n = \omega$ has n distinct complex roots that lie on the circle, according to the Moivre's formula;
- 3) *f(z)=g(z)*: the method of equating real and imaginary parts is one of the basic and effective ways of solving the equations in the set C .

Example 5.9. Calculate the value of an expression

$$z^{167} + \frac{1}{z^{167}}$$

for z , which satisfies the equation $z + \frac{1}{z} = 1$.

Solution. By the condition z satisfies the equation $z^2 - z + 1 = 0$, i. e. $z^3 = -1$.
Since

$$z^{167} = (z^3)^{55} \cdot z^2 = -z^2,$$

then you need to calculate the value of the expression

$$-z^2 - \frac{1}{z^2} = -\left(z^2 + \frac{1}{z^2}\right).$$

Let's square the equation $z + \frac{1}{z} = 1$ and find that

$$z^2 + \frac{1}{z^2} = -1.$$

Hence, $z^{167} + \frac{1}{z^{167}} = 1$.

5.6. Some well-known inequalities

1. $\forall a, b \in R : a^2 + b^2 \geq 2ab$.

2. If a i $b \in R$ have the same sign (i.e. $ab > 0$), then

$$\frac{a}{b} + \frac{b}{a} \geq 2.$$

3. If $a \geq 0$ i $b \geq 0$, then $\frac{a+b}{2} \geq \sqrt{ab}$, that is, the arithmetic value of two non-negative numbers is not less than their geometric value.

4. Inequality for the arithmetic mean and geometric mean of n non-negative numbers (Cauchy inequality): *non-negative numbers*,: if $a_1, a_2, \dots, a_n - \forall$ non-negative numbers, then

$$\left| \frac{a_1 + a_2 + \dots + a_n}{n} \right| \geq \sqrt[n]{a_1 \cdot a_2 \cdot \dots \cdot a_n}.$$

Equality holds when $a_1 = a_2 = \dots = a_n$.

5. **Cauchy-Bunyakovsky** inequality: for any real numbers $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ the next inequality takes place:

$$(a_1 b_1 + a_2 b_2 + \dots + a_n b_n)^2 \leq (a_1^2 + a_2^2 + \dots + a_n^2)(b_1^2 + b_2^2 + \dots + b_n^2).$$

Equality takes place if and only if the numbers a_k i b_k are proportional, that is, when there are such numbers α and $\beta, \alpha^2 + \beta^2 \neq 0$ and for all $k = 1, 2, \dots, n$ the equality $\alpha a_k + \beta b_k = 0$ holds.

6. **Bernoulli's** inequality: if $x \geq -1$, then for any $n \in N$ the inequality $(1 + x)^n \geq 1 + nx$ holds.

7. **Hölder's** inequality: for any non-negative real number pairs $(x_i; y_i)$, where $i = 1, n$ and the positive rational numbers $p, q: \frac{1}{p} + \frac{1}{q} = 1$, the inequality takes place:

$$\begin{aligned} & (x_1^p + x_2^p + x_3^p + \dots + x_n^p)^{\frac{1}{p}} \cdot (y_1^q + y_2^q + y_3^q + \dots + y_n^q)^{\frac{1}{q}} \\ & \geq (x_1 y_1 + x_2 y_2 + x_3 y_3 + \dots + x_n y_n) \end{aligned}$$

8. **Minkovski** inequality:

$$(x_1^p + x_2^p + \dots + x_n^p)^{\frac{1}{p}} + (y_1^p + y_2^p + \dots + y_n^p)^{\frac{1}{p}} + \dots + (z_1^p + z_2^p + \dots + z_n^p)^{\frac{1}{p}} \geq ((x_1 + y_1 + \dots + z_1)^p + (x_2 + y_2 + \dots + z_2)^p + \dots + (x_n + y_n + \dots + z_n)^p)^{\frac{1}{p}}.$$

9. **Chebyshev** inequality: for two non-decreasing (or two non-increasing) sequences of numbers a_i and b_i the inequality holds:

$$\frac{a_1 + a_2 + a_3 + \dots + a_n}{n} \cdot \frac{b_1 + b_2 + b_3 + \dots + b_n}{n} \leq \frac{a_1 b_1 + a_2 b_2 + a_3 b_3 + \dots + a_n b_n}{n}.$$

Example 5.10. Find all m for which the inequality

$$mx^2 - 4x + 3m + 1 > 0$$

is satisfied for all positive values x .

Solution. The condition that the branches of the parabola are directed upwards means that $m > 0$. If the parabola does not intersect the Ox axis, then we obtain the system

$$\begin{cases} m > 0, \\ D = 4 - m(3m + 1) < 0. \end{cases}$$

If the given trinomial has the real roots, then the larger root must not be positive:

$$\begin{cases} m > 0, \\ \frac{2 + \sqrt{4 - m(3m + 1)}}{m} \leq 0. \end{cases}$$

The second inequality of the second system (and, as a consequence, the entire system) has no the solutions when $m > 0$, since the numerator and denominator are positive.

Solving the second inequality of the first system, we obtain:

$$m < -1\frac{1}{3}, m > 1.$$

Taking into account the first inequality, we find the solution of the inequality: $m > 1$.

Let's now $m=0$. Then we obtain, that $-4x + 1 > 0$, i. e. $x < \frac{1}{4}$ and the inequality is not satisfied for all positive x .

Hence, $m > 1$.

5.7. Application of the derivative and integral when proving inequalities

When proving inequalities using the derivative, the following considerations are made [6], [9].

Let us assume that on the interval $x \in [a, b]$ from the domain of the functions f and g we need to prove the inequality $f(x) \geq g(x)$. Consider the function $u(x) = f(x) - g(x)$. Let the derivative $u'(x)$ have a single root x_0 on a given interval, x_0 is the minimum point of the function $u(x)$ and the equality $u(x_0) = f(x_0) - g(x_0)$ holds. This is enough to prove that on the interval $[a, b]$ the inequality $f(x) \geq g(x)$ holds.

This method can also be used to prove the numerical inequalities. Then, before implementing the scheme described above, consider a function that has the numerical values at some points.

Example 5.11. Prove the inequality $\cos x \geq 1 - \frac{x^2}{2}$ for any real x .

Proof. Let's have the function $f(x) = \cos x - 1 + \frac{x^2}{2}$. The derivative of a function $f'(x) = x - \sin x$ has the unique root $x=0$. Hence, $x=0$ is the minimum point of the function. So, for all values $x \neq 0$ the inequality will hold $f(x) = \cos x - 1 + \frac{x^2}{2} > 0$. The equality is achieved when $x=0$.

When proving the inequalities using an integral number, the following measurement is achieved [6], [9]. Let two continuous functions $f(x)$ and $g(x)$ be given on the interval $[a, b]$ and at all points of this interval the inequality $f(x) \geq g(x)$ holds. Then on the interval $[a, b]$ the inequality $\int_a^x f(t) dt \geq \int_a^x g(t) dt$ holds. Similarly, the statement holds for the following functions $f(x) > g(x)$, $f(x) \leq g(x)$ and $f(x) < g(x)$.

The procedure for implementing this method is as follows: to prove the inequality $F(x) \geq G(x)$ consider the functions $f(x)$ та $g(x)$, where $F'(x) = f(x)$, $G'(x) = g(x)$. When performing the inequality $f(x) \geq g(x)$, we claim that the inequality $F(x) = \int_a^x f(t) dt \geq G(x) = \int_a^x g(t) dt$ is also true.

Example 5.12. Prove the inequality

$$\sqrt{2} + \sqrt{3} + \dots + \sqrt{2019} < \frac{8080\sqrt{505}-4\sqrt{2}}{3}.$$

Solution. Consider the function $f(x) = \sqrt{x}$. Each term $\sqrt{n}$ can be interpreted as the area of a rectangle with height $\sqrt{n}$ and base equal to 1 (the distance between points n and $n + 1$), then

$$\begin{aligned} \sqrt{2} + \sqrt{3} + \dots + \sqrt{2019} &< \int_2^{2020} \sqrt{x} dx = \\ &= \frac{2}{3} \cdot x^{\frac{3}{2}} \Big|_2^{2020} = \frac{2\sqrt{2020^3}-2\sqrt{2^3}}{3} = \frac{8080\sqrt{505}-4\sqrt{2}}{3}. \end{aligned}$$

Control questions

1. What is the canonical form of a polynomial of degree n ?
2. What operations are performed on polynomials?
3. What concepts of higher Algebra from the theory of polynomials are studied in the school course of Mathematics?
4. What is an algebraic equation?
5. Formulate Bezu's theorem and the fundamental theorem of algebra.
6. Give the different approaches to the definition of the concept of equation (inequality).
7. What transformations of equations (inequalities) are equivalent?
8. Formulate the definitions of equations with a parameter and two parameters and indicate what are the methods for solving them.
9. What are the methods for solving equations in the domain of complex numbers?
10. Give some well-known inequalities (mean inequalities, Cauchy, Cauchy-Buniakovsky, Bernoulli, Hölder, Minkovsky, Chebyshev).

REFERENCES

1. Barbeau E. J. Polynomials. Springer-Verlag New-York Inc., 1989. 441 p.
2. Charles C. Pinter . A Book of Set Theory. Kyung Moon Publishers, South Korea, 2014. 224 p.
3. Gal Yu. M., Zolota O. A. Projective Geometry and Foundations of Geometry. Drohobych: RVDDPU, 2011. 125 p. [in Ukrainian]
4. Herman J., Šimša J., Kučera R. Equations and Inequalities. Elementary Problems and Theorems in Algebra and Number Theory. Canadian Mathematical Society, 2000. 344 p.
5. History of the development of Mathematics. [in Ukrainian] Available at: <https://surl.li/kwfhbw>
6. Kutnyak O. Scientific Foundations of the School Course of Mathematics. Part I. Drohobych: DIVSPU, 2026. 62 p. [in Ukrainian]
7. Morphism. Available at: <https://en.wikipedia.org/wiki/Morphism>
8. Nicholas J., Hunter J., Hargreaves J. Functions and Their Graphs. Mathematics Learning Centre, University of Sydney, 2006. 78 p.
9. Sobkovych R., Kulchytska N. Basic methods of proving inequalities. Ivano-Frankivsk, 2014. 100 p. [in Ukrainian]
10. Sokolenko L. O. Scientific foundations of the school mathematics course. Part I. Chernihiv: Desna Polygraph, 2020. 144 p. [in Ukrainian]
11. Topological space. Available at: https://en.wikipedia.org/wiki/Topological_space

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